Uncertainty Aversion with Multiple Issues

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Abstract

We study a decision problem under uncertainty about multiple issues by explicitly imposing a product structure on the set of states in the Anscombe-Aumann framework. In this environment, a decision maker may exhibit a tendency to avoid uncertain acts that depend on many issues since it can be harder to form a belief about multiple issues jointly than about individual issues separately. We provide a novel behavioral property, *Multi-Issue Uncertainty Aversion*, which captures this idea. The property blends two concepts of aversion to uncertainty. First, it requires that when there are pairwise indifferent acts that depend on distinct issues, a mixture of them that demands multi-issue considerations must be less preferred to each individual act. Second, the property imposes the Uncertainty Aversion axiom of Gilboa and Schmeidler (1989) among the alternatives that depend on a single issue. We characterize the set of utility functions consistent with Multi-issue Uncertainty Aversion within the broad class of invariant biseparable preferences. We show that exhibiting Multi-Issue Uncertainty Aversion is equivalent to having a belief satisfying two conditions: *exhaustiveness of the core* of the (joint) belief and *superadditivity* of the marginal beliefs. The exhaustiveness condition provides a novel way of comparing a decision maker's degrees of uncertainty aversion about different sets of issues.

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1 Introduction

Motivation. Many decision problems involve uncertainty about multiple issues. An investor selecting a portfolio faces uncertainty about the returns of different securities, and an analyst making a forecast must consider many aspects of a firm that will evolve over time. It is both natural and common that people divide future uncertainty into several issues either because a future event is just a combination of outcomes regarding each issue or because the division provides a more convenient way to think about future events. Regardless of the reason, once they do separate the issues, people are likely to form a belief about future events based on the relevant issues. In other words, they first think about the likelihoods of outcomes regarding individual issues separately and then go on to consider the relationship between issues—for example, whether the returns of securities considered are positively or negatively correlated.

The process of uncovering the relationship or correlation across issues adds significant challenges to the entire belief-formation process. One reason is the possibility of insufficient information. For example, an investor trying to learn the correlation between the returns of various securities from past data would need frequent simultaneous observations of the returns, which may not exist.[1] Another reason is computational complexity. Even if a sizable amount of data is gleaned, deriving a sophisticated estimation from them may not be easy, especially when a large number of issues are involved.[2]

In light of this difficulty, more issues being involved may mean a larger amount of uncertainty, which an uncertainty averse decision maker would try to avoid. In other words, when there are two options and one of them depends on a smaller set of issues than the other, the decision maker may select the former to reduce the amount of uncertainty she faces. This paper formalizes a concept of uncertainty aversion in this sense.

The following thought experiment makes our idea concrete. Suppose there is an urn which contains 100 balls in it. Each ball is colored, either yellow (Y) or blue (B), and marked with a letter, either L or R, but the exact composition of the urn is unknown. A ball is to be randomly drawn from the urn. There are three options a subject is asked to rank before a ball is drawn. The options are depicted in Figure 1. The two rows, Y and B, represent the color of the ball, and the two columns, L and R, the letter marked on the ball. The number in each cell is the probability of the subject receiving a good prize, \$100. For example, if the subject chooses option Y, then she receives \$100 for sure if the ball is yellow

[1] Aït-Sahalia et al. (2010) and Zhang (2011) discuss the challenges in estimation with asynchronous data.

[2] Various challenges in estimating high-dimensional models including correlations are documented in the literature, for example, by Chan et al. (1999) and Chib et al. (2006). See also surveys by Andersen et al. (2006) and Bauwens et al. (2006).

	L	R
Y	1	1
B	0	0

Option Y

	L	R
Y	1	0
B	1	0

Option L

	L	R
Y	1	0.3
B	0.7	0

Option M

Figure 1: The options considered in the thought experiment: The numbers in each cell represent the probabilities of a subject receiving a good prize, \$100.

and \$0 for sure if it is blue (regardless of its letter).

A plausible ranking over these options is

$$Y \sim L \succ M\,. \tag{1}$$

A subject may explain these preferences by the following argument. First, due to the principle of insufficient reason, the subject may deem the events equally likely and thus be indifferent between betting on colors and betting on letters, which leads to $Y \sim L$. Second, our argument for the strict parts ($Y \succ M$ and $L \succ M$) is as follows. Both options Y and L depend only on one issue, either colors (option Y) or letters (option L). However, if the subject chooses option M, she will be exposed to the uncertainty about colors, letters, and the correlation between them. To avoid this additional uncertainty, the subject may rank options Y and L over M, which results in the order in (1).

There are two interesting aspects about the illustrated ranking. First, it is inconsistent with Subjective Expected Utility (Anscombe and Aumann, 1963). In other words, whatever probability distribution over YL, YR, BL, and BR an expected utility maximizer holds, she will *not* exhibit the ranking in (1). Since option M is a mixture of options Y and L (with weights 0.3 on Y and 0.7 on L), an expected utility maximizer would rank option M between Y and L, which is not the case in (1).

Second, even though we motivated the ranking as a behavior that captures a desire to avoid uncertainty, it is also inconsistent with the *Uncertainty Aversion* axiom of Gilboa and Schmeidler (1989; GS henceforth). Under their assumption, option M must be weakly preferred to options Y and L because it is a mixture of those two indifferent options.[3] GS asserts that a mixture of alternatives gives a hedging opportunity, and uncertainty averse behavior is characterized by the preference for the hedging effect. However, if the alternatives to be mixed depend on distinct issues and a decision maker poorly understands the relationship between the issues, such a hedging effect is likely to be marginal at best. Rather, some mixtures, such as option M, may increase the number of uncertain dimensions, thus becoming a

[3] We provide the formal definition of the Uncertainty Aversion axiom in Section 4 (Definition 3).

more uncertain alternative. Our thought experiment illustrates that when a decision maker is concerned about the number of issues relevant to each alternative, the choices she makes to avoid uncertainty may directly conflict with the prominent concept of Uncertainty Aversion that GS provides. As the multi-issue structure is common, natural, and widely used in the economics literature, this behavior pattern merits a thorough exploration.

Preview of main results. Our primary goal in this paper is to formalize the novel behavioral concept of uncertainty aversion illustrated in our thought experiment, and then to examine what condition on utility functions is consistent with such behavior. We answer these questions using the framework of Anscombe and Aumann (1963), in which a decision maker chooses among acts—mappings from states to lotteries. Crucially, on top of the standard model, we additionally assume that the set of states is a product set, reflecting the multi-issue environment. Then, in Section 4, we define a behavioral property called *Multi-Issue Uncertainty Aversion (MIUA)*. It is meant to impose two main behavioral restrictions. One requires that when there are pairwise indifferent acts that depend on distinct issues, a mixture of them, which demands multi-issue considerations, must be less preferred to each individual act. This restriction captures the behavior manifested by the ranking (1) in our thought experiment. The other restriction imposes the Uncertainty Aversion axiom of GS among acts that depend on a *single* issue. Even though mixtures are not globally preferred under MIUA, it is still desirable when only one issue matters and hence a decision maker is not concerned about an increase in the number of relevant issues. Thus, MIUA is viewed as an extension of the Uncertainty Aversion axiom of GS to a multi-issue environment.

To characterize the set of utility functions consistent with MIUA, we focus on the class of *invariant biseparable preferences* proposed by Ghirardato, Maccheroni, and Marinacci (2004; GMM henceforth) and studied by Gilboa et al. (2010) and Chandrasekher et al. (2020) among others. The class is broad enough to include Subjective Expected Utility, Choquet Expected Utility (Schmeidler, 1989), and Maxmin Expected Utility preferences (GS). The latter is achieved by imposing the Uncertainty Aversion axiom on the invariant biseparable preference.[4] A nice property this class provides is that we can separate a decision maker's preference over lotteries (represented by a utility index) and her belief about states (represented by a belief functional). In particular, her risk attitude is fully captured by the former, while her uncertainty attitude—which is our main focus here—is captured by the

[4]Of course, some papers study uncertainty aversion without assuming the axioms of GMM. The variational preference (Maccheroni et al., 2006) is derived by adding the Uncertainty Aversion axiom to a set of axioms weaker than those of the invariant biseparable preference. Ghirardato and Siniscalchi (2012) also analyze a class of preferences that subsumes invariant biseparable preferences.

latter.

In our characterization, we use the concept of a core that is widely used in the literature about uncertainty aversion (Ghirardato and Marinacci, 2002; Chateauneuf and Tallon, 2002; Grant and Polak, 2013). Our main result shows that MIUA is equivalent to the combination of two conditions: *exhaustiveness* of the core of a joint belief and *superadditivity* of marginal beliefs (Theorem 1). While the latter directly comes from GS's Uncertainty Aversion imposed on each single issue, the exhaustiveness is a novel condition we provide in this paper. This condition compares the core of a decision maker's joint belief to the cores of her marginal beliefs about individual issues. The exhaustiveness condition is satisfied when the former is sufficiently large relative to the latter. It can be interpreted as the decision maker being more averse to uncertainty about the entire set of issues collectively than to uncertainty about individual issues separately. This interpretation is exactly how we motivate the ranking (1) in our thought experiment.

Related literature. Starting with the seminal work of Ellsberg (1961), the notion of uncertainty (or ambiguity) aversion has been widely studied. Schmeidler (1989) introduces non-additive probability (or capacity) and provides the foundation of the Choquet Expected Utility. This generalization of Subjective Expected Utility can accommodate the behavior in Ellsberg (1961) along with the Uncertainty Aversion axiom (See also Wakker (1990) for an alternative formulation). This axiom is also used to establish the axiomatic foundations of the Maxmin Expected Utility (Gilboa and Schmeidler, 1989) and the variational preference (Maccheroni, Marinacci, and Rustichini, 2006; Strzalecki, 2011). As we already noted, we will extend this axiom to a multi-issue environment in a way that preference for a mixture is not global.

Other notions of uncertainty aversion have been discussed in the literature as well. Epstein (1999) and Ghirardato and Marinacci (2002) provide intuitive definitions that are analogous to that of risk aversion. Chateauneuf and Tallon (2002) weakens the Uncertainty Aversion axiom of GS to propose Preference for Sure Diversification. Our paper is closely related to Ghirardato and Marinacci (2002) and Chateauneuf and Tallon (2002) in that the characterization of utility functions is achieved through the concept of a core. Moreover, the property MIUA is shown to be equivalent to their axioms under some circumstances, which we discuss in Section 5 (Theorem 3). However, we demonstrate a novel way of using a core to compare a decision maker's different degrees of aversion to uncertainty about different sets of issues. This contrasts with Ghirardato and Marinacci (2002) who compare degrees of uncertainty aversion of two different decision makers.

In our comparison of degrees of aversion, a decision maker exhibiting MIUA can be understood to be less averse to one issue than to multiple issues. In this sense, our notion of an i-act, which depends only on one issue i, is similar to an unambiguous act defined in Epstein (1999) and Epstein and Zhang (2001). However, while the preference over unambiguous acts in their models are required to be probabilistically sophisticated, MIUA in our paper still allows a decision maker to suffer from ambiguity and have a Maxmin Expected Utility preference even among i-acts. The case when the decision maker has an expected-utility preference over i-acts is discussed as a special case of our model in Section 5.

Models with a product set of states have been studied in many papers. Walley and Fine (1982), Hendon et al. (1996), and Ghirardato (1997) study capacities on product sets with implications for the Choquet Expected Utility preference. GS also studies the Maxmin Expected Utility with a product set of states. Some of the preferences proposed in these papers will be discussed in Section 6 and shown to be consistent with MIUA. Ergin and Gul (2009) also study a model with two-dimensional states in the Savage (1972) framework. They call each dimension an issue, the term we adopt in this paper. Unlike ours, however, their paper regards different issues as different stages of a compound lottery. A decision maker in their model has a probabilistic belief about each issue, treats two issues as independent of each other, and evaluates an act in an iterative manner by resolving one issue after another as if she has a second-order belief (Klibanoff et al., 2005; Seo, 2009). In other words, two issues in their model have a vertical relationship as the risk about lotteries and the uncertainty about states do in the Anscombe and Aumann framework. The issues in our model, however, are parallel.

In addition to these theoretical analyses, Epstein and Halevy (2019) run a laboratory experiment that incorporates two-dimensional states. Their experiment is very closely related to our thought experiment, and the underlying ideas are similar. However, our theoretical analysis presented in this paper does not directly address their experimental observations. While we study the GS type mixture of acts that entangles multiple issues, the options in their experiment are understood to be generated by a different type of mixture. We will elaborate more on this point in Section 7. We believe that studying both types of behavior is vital in understanding uncertainty aversion in multi-issue environments.

Our paper is related to the literature on correlation misperception and correlation neglect in that we consider a situation in which correlations between multiple issues are poorly understood. Ellis and Piccione (2017) study a model in which a decision maker correctly understands what outcomes can be realized from each alternative, but has an incorrect understanding of what joint realizations of outcomes can be obtained from multiple alternatives (correlation misperception). Kochov (2018) focuses on the preferences that arise from misunderstanding the autocorrelations of outcomes across periods in a dynamic model. Levy and

Razin (2022) study how a forecast about a multi-dimensional state can be made by combining multiple sources of information when the correlations between them are unknown. They show that under certain conditions, a forecaster may make a prediction as if there is no correlation at all (correlation neglect). Recent experimental studies also suggest that people often neglect correlations between asset returns (Eyster and Weizsäcker, 2016) and information sources (Enke and Zimmermann, 2019). While these papers study the implications of subjectively assuming an incorrect correlation different from the true or objective one, we focus on the behavior that favors an alternative whose outcome distribution is independent of the unknown correlation.

Last but not least, our analysis has implications for under-diversification observed in financial markets (Dow and Werlang, 1992; Cao et al., 2005; Easley and O'Hara, 2009, Boyle et al., 2012; Gorton and Metrick, 2013). In particular, the property MIUA provides a way to directly impose anti-diversifying behavior, and the utility functions presented in Section 6 can be used to represent such behavior. We will discuss more in this regard in Section 7.

Organization of the paper. The rest of the paper is organized as follows. We present our model in Section 2 and introduce the invariant biseparable preference in Section 3. In Section 4, we define MIUA and present our main result, characterizing the utility functions consistent with MIUA. Section 5 discusses a special case with additive marginal beliefs and shows that MIUA can be characterized more simply in that case. Then, we discuss examples of utility functions in Section 6 and our model's implications for under-diversification in financial markets and relationship to other experimental studies in Section 7. Concluding remarks follow in Section 8.

2 Model

2.1 Primitives

Let $\mathcal{I}$ be a nonempty finite set. Each element in $\mathcal{I}$ is called an *issue*. For each issue $i \in \mathcal{I}$, S_i is a nonempty finite set, whose elements encode complete descriptions of all relevant consequences regarding issue i. A *state* is an element of the product set $S = \times_{i\in\mathcal{I}} S_i$. A subset of S is called an *event*.

A nonempty set $\mathcal{Z}$ is the set of *outcomes*. The set of all simple lotteries, or probability measures with finite supports, over $\mathcal{Z}$ is denoted by $\mathcal{L}$. An *act* is a function $f : S \to \mathcal{L}$, which prescribes which lottery will be delivered at each future state. We denote the set of all acts by $\mathcal{F}$. A decision maker's (DM henceforth) *preference*, denoted by $\succsim$, is a complete and transitive binary relation over $\mathcal{F}$. As usual, $\succ$ and $\sim$ denote the asymmetric and symmetric parts of $\succsim$, respectively.

2.2 Terminology and notation

In this subsection, we provide some terminology and notation that will be used throughout this paper. Given the product structure of the set of states, some events may be independent of a certain set of issues. When we can tell whether an event occurs or not only by looking at issues in $J \subset \mathcal{I}$, the event is called a J-event. Likewise, when the final lottery delivered by an act can be completely determined by issues in $J \subset \mathcal{I}$, the act is called a J-act. Formally, for a nonempty subset $J \subset \mathcal{I}$, an event $E \subset S$ is called a *J-event* if $E = E_J \times S_{J^c}$ for some $E_J \subset S_J$, where S_K denotes the Cartesian product $\times_{i \in K} S_i$ for each $K \subset \mathcal{I}$. The collection of all J-events forms an algebra $\mathcal{A}_J$ on S. Given this, an act is a *J-act* if it is measurable with respect to $\mathcal{A}_J$. We denote the set of all J-acts by $\mathcal{F}_J$. We use an analogous notation $\mathcal{F}_\varnothing$ to denote the set of all constant acts. We say that an issue i is *irrelevant* to an act if the act belongs to $\mathcal{F}_{\mathcal{I} \setminus \{i\}}$. Otherwise, issue i is *relevant* to the act.

For each nonempty subset $J \subset \mathcal{I}$, $\Delta(S_J)$ denotes the set of all probabilities on S_J. We identify $\Delta(S_J)$ as a subset of $\mathbb{R}^{S_J}$ with the natural embedding. Given a probability $P \in \Delta(S)$, $marg_J(P) \in \Delta(S_J)$ denotes the marginal probability of P on S_J. The *marginal preference relation of* $\succsim$ *on* $\mathcal{F}_J$ is the restriction of the preference relation on $\mathcal{F}_J$. We denote it by $\succsim|_J$.[5]

We use the usual notation for mixed acts. For acts $f_1, \cdots, f_n \in \mathcal{F}$ and weights $\alpha_1, \cdots, \alpha_n \in [0,1]$ with $\sum_{k=1}^n \alpha_k = 1$, the mixed act $\alpha_1 f_1 + \cdots + \alpha_n f_n = \sum_{k=1}^n \alpha_k f_k \in \mathcal{F}$ is the statewise mixture of $f_1, \cdots, f_n$, that is, $\big(\sum_{k=1}^n \alpha_k f_k\big)(s) = \sum_{k=1}^n \alpha_k f_k(s)$ for each $s \in S$.

For our analysis, it will be convenient to introduce the concept of a utility act. A *utility act* is a vector $v \in \mathbb{R}^S$, where $v(s) \in \mathbb{R}$ is interpreted as the utility level achieved at state s. For example, given a real-valued (utility) function $u : \mathcal{L} \to \mathbb{R}$ and an act $f : S \to \mathcal{L}$, the composition of u and f, denoted by $u(f)$, is a utility act. In this case, $u(f)$ represents the utility level attained at each state by choosing the act f.

Lastly, we define two mappings that change dimensions of certain objects. First, for a lottery $x \in \mathcal{L}$, $\bar{x} \in \mathcal{F}_\varnothing$ denotes the constant act that delivers x at all states. Similarly, for a number $c \in \mathbb{R}$, $\bar{c} \in \mathbb{R}^S$ denotes the constant utility act with $\bar{c}(s) = c$ for all $s \in S$. Second, for a J-act $f \in \mathcal{F}_J$, $\varphi_J(f)$ is the natural transformation of f into a mapping from S_J to $\mathcal{L}$. That is, $\varphi_J(f)$ satisfies $\varphi_J(f)(s_J) = f(s_J, s_{-J})$ for all $s_J \in S_J$ and $s_{-J} \in S_{J^c}$. In addition, for a utility act $v \in \mathbb{R}^S$ measurable with respect to $\mathcal{A}_J$, $\varphi_J(v)$ denotes the similar transformation

[5] Formally, $\succsim|_J = \succsim \cap (\mathcal{F}_J \times \mathcal{F}_J)$.

of v into a vector in $\mathbb{R}^{S_J}$.[6]

3 Invariant Biseparable Preferences

In this section, we introduce the class of *invariant biseparable preferences* which will be discussed throughout this paper. They are widely studied in the literature, for example, by GS, GMM, Gilboa et al. (2010), Chandrasekher et al. (2020) among others.[7]

3.1 Invariant biseparable representation

We define an invariant biseparable preference as a preference that admits what we call *invariant biseparable representation*. The representation is a generalization of Subjective Expected Utility (SEU). An SEU preference allows a representation $P \cdot u(f)$, where $u : \mathcal{L} \to \mathbb{R}$ is an affine utility index and P is a probability on S. An SEU maximizer behaves as if she converts an act $f \in \mathcal{F}$ into a utility act $u(f) \in \mathbb{R}^S$ using the utility index u, and then computes the expectation of $u(f)$ with respect to P. The latter part maps the utility act $u(f) \in \mathbb{R}^S$ to a final utility level $P \cdot u(f) \in \mathbb{R}$, and this involves the decision maker's belief over states. In light of this, we call the mapping $u(f) \mapsto P \cdot u(f)$ the belief functional of the SEU representation.

The invariant biseparable representation allows a more general form of belief functionals. Let $\mathbb{B} : \mathbb{R}^S \to \mathbb{R}$ be a functional. We say $\mathbb{B}$ is *monotonic* if $v(s) \geq w(s)$ for all $s \in S$ implies $\mathbb{B}(v) \geq \mathbb{B}(w)$. It is *constant linear* if $\mathbb{B}(v + \bar{c}) = \mathbb{B}(v) + c$ and $\mathbb{B}(\alpha v) = \alpha \mathbb{B}(v)$ for all $v \in \mathbb{R}^S$, $c \in \mathbb{R}$, and $\alpha \in \mathbb{R}_+$. It can be easily seen that an SEU belief functional is both monotonic and constant linear. From now on, we only discuss belief functionals that are monotonic and constant linear. The following is the formal definition of the invariant biseparable representation and preference.

Definition 1. A pair $(u, \mathbb{B})$ consisting of a nonconstant affine utility index $u : \mathcal{L} \to \mathbb{R}$ and a monotonic and constant linear belief functional $\mathbb{B} : \mathbb{R}^S \to \mathbb{R}$ is an *invariant biseparable representation* of the preference relation $\succsim$ if the utility function $f \mapsto \mathbb{B}(u(f))$ represents $\succsim$. The preference relation $\succsim$ is an *invariant biseparable preference* if it has an invariant biseparable representation.

[6] Let $\mathcal{R}_J$ denote the set of utility acts measurable with respect to $\mathcal{A}_J$:

$$\mathcal{R}_J = \left\{ v \in \mathbb{R}^S : v(s_J, s_{-J}) = v(s_J, s'_{-J}), \forall s_J \in S_J, \forall s_{-J}, s'_{-J} \in S_{J^c} \right\}.$$

Then, for each $v \in \mathcal{R}_J$, $\varphi_J(v)(s_J) = v(s_J, s_{-J})$ for all $s_J \in S_J$ and $s_{-J} \in S_{J^c}$. The function φ_J maps $\mathcal{F}_J \cup \mathcal{R}_J$ to $\mathcal{L}^{S_J} \cup \mathbb{R}^{S_J}$. Note that φ_J is a bijection and the inverse is well-defined.

[7] The name 'invariant biseparable preferences' is given by GMM.

An axiomatic foundation for the invariant biseparable representation is provided by GMM, which we introduce in Appendix A. For the most part of this paper, we will assume that the preference relation $\succsim$ is an invariant biseparable preference. By doing so, we can separate the DM's preference over lotteries represented by a utility index u and her belief over states represented by a belief functional $\mathbb{B}$ which is uniquely identified (Proposition A.1). While the former captures the DM's risk attitude, the latter captures her uncertainty attitude. As is often the case with papers about uncertainty aversion, our analysis will be primarily focused on the characterization of the set of belief functionals consistent with the behavioral property we will provide in later sections. As is shown in Example 1 below, SEU (Anscombe and Aumann, 1963), Choquet Expected Utility (CEU) (Schmeidler, 1989), and Maxmin Expected Utility (MEU) of GS are well known examples of the invariant biseparable representation with different specifications of monotonic and constant linear belief functionals.

Example 1. *Let $u : \mathcal{L} \to \mathbb{R}$ be a nonconstant affine utility index.*

(1) The SEU representation is given by $P \cdot u(f)$, where $P \in \Delta(S)$. The belief functional is $\mathbb{B}(v) = P \cdot v$ for each $v \in \mathbb{R}^S$.

(2) The CEU representation is given by $\int_S u(f)\, d\nu$, where ν is a capacity on S and the integral is a Choquet integral.[8] The belief functional is $\mathbb{B}(v) = \int_S v\, d\nu$.

(3) The MEU representation is given by $\min_{P\in C} P \cdot u(f)$, where $C \subset \Delta(S)$ is nonempty, closed, and convex. The belief functional is $\mathbb{B}(v) = \min_{P\in C} P \cdot v$.

□

We conclude this subsection by briefly discussing the two properties, invariance and biseparability, from which the name of the preference originates. Suppose $(u, \mathbb{B})$ is an invariant biseparable representation of $\succsim$. First, biseparability means that there exists a monotonic set function, or a capacity, $\nu : 2^S \to [0, 1]$ such that for any lotteries $x, y \in \mathcal{L}$ with $\bar{x} \succsim \bar{y}$ and any event $E \subset S$, the binary act $\bar{x}E\bar{y}$ has a utility level

$$\mathbb{B}\big(u(\bar{x}E\bar{y})\big) = \nu(E)u(x) + (1 - \nu(E))u(y)\,. \tag{2}$$

Such a set function can be found by taking $\nu(E) = \mathbb{B}(e_E)$, where $e_E \in \mathbb{R}^S$ is an indicator

[8] See Appendix C for more details about the Choquet Expected Utility.

function such that $e_E(s) = 1$ for all $s \in E$ and $e_E(s) = 0$ for all $s \notin E$.[9] Given the equation (2), the value $\mathbb{B}(e_E)$ can be interpreted as the likelihood the DM assigns to the event E. Second, invariance means that even if we choose a different normalization of the utility index, say u', the same belief functional $\mathbb{B}$ still represents $\succsim$ along with u'. That is, for any $a > 0$, $b \in \mathbb{R}$, and for any acts $f, g \in \mathcal{F}$,

$$\mathbb{B}(u(f)) \geq \mathbb{B}(u(g)) \quad \Longleftrightarrow \quad \mathbb{B}(au(f) + \bar{b}) \geq \mathbb{B}(au(g) + \bar{b})\,. \tag{3}$$

This invariance is an immediate consequence of imposing constant linearity on the belief functional $\mathbb{B}$. We refer the interested readers to Ghirardato and Marinacci (2001, 2002) and GMM for more details.

3.2 Marginal belief

In this subsection, we define a marginal belief functional which will play an important role in our analysis. Suppose $(u, \mathbb{B})$ is an invariant biseparable representation of the preference relation $\succsim$, and let $J \subset \mathcal{I}$ be a nonempty subset. Roughly speaking, a marginal belief functional with respect to J is a belief functional defined on $\mathbb{R}^{S_J}$, which, jointly with the utility index u, represents the marginal preference relation $\succsim|_J$. To formalize, let $\mathcal{R}_J$ be a subspace of $\mathbb{R}^S$ that includes all utility acts measurable with respect to the algebra $\mathcal{A}_J$:

$$\mathcal{R}_J = \left\{ v \in \mathbb{R}^S : v(s_J, s_{-J}) = v(s_J, s'_{-J})\,, \forall s_J \in S_J, \forall s_{-J}, s'_{-J} \in S_{J^c} \right\}. \tag{4}$$

It is noteworthy that for any J-act $f \in \mathcal{F}_J$, the corresponding utility act $u(f)$ belongs to $\mathcal{R}_J$. A marginal belief functional is defined as follows.

Definition 2. Let $(u, \mathbb{B})$ be an invariant biseparable representation of the preference relation $\succsim$, and let a nonempty subset $J \subset \mathcal{I}$ be given. The *marginal belief functional of* $\succsim$ *with respect to* J is the functional $\mathbb{B}_J : \mathbb{R}^{S_J} \to \mathbb{R}$ defined by

$$\mathbb{B}_J(v) = \mathbb{B}\big(\varphi_J^{-1}(v)\big)\,, \quad \forall v \in \mathbb{R}^{S_J}\,.$$

In other words, a marginal belief functional $\mathbb{B}_J$ is the restriction of $\mathbb{B}$ to $\mathcal{R}_J$ *identified as*

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[9] Since $u(\bar{x}E\bar{y}) = (u(x) - u(y))e_E + \overline{u(y)}$, constant linearity of $\mathbb{B}$ implies that

$$\begin{aligned}\mathbb{B}\big(u(\bar{x}E\bar{y})\big) &= \mathbb{B}\big((u(x) - u(y))e_E + \overline{u(y)}\,\big)\\ &= (u(x) - u(y))\mathbb{B}(e_E) + \mathbb{B}\big(\overline{u(y)}\,\big)\\ &= \mathbb{B}(e_E)u(x) + (1 - \mathbb{B}(e_E))u(y)\,.\end{aligned}$$

Rewriting $\mathbb{B}(e_E) = \nu(E)$, we obtain (2). The set function ν inherits monotonicity from $\mathbb{B}$.

a functional on $\mathbb{R}^{S_J}$. Hence, it is straightforward that $\mathbb{B}_J$, as well, is monotonic and constant linear. We can view $(u, \mathbb{B}_J)$ as an invariant biseparable representation of $\succsim|_J$. The utility level of a J-act $f \in \mathcal{F}_J$ under the representation is given by

$$\mathbb{B}_J\big(u(\varphi_J(f))\big) = \mathbb{B}_J\big(\varphi_J(u(f))\big)\,,$$

which is, of course, equal to $\mathbb{B}(u(f))$.

Example 2. *Consider an SEU representation with a utility index u and a probability P. For any J-act $f \in \mathcal{F}_J$, we have*

$$P \cdot u(f) = marg_J(P) \cdot u(\varphi_J(f))$$

and we can see that $v \mapsto marg_J(P) \cdot v$ is the marginal belief functional with respect to J. Thus, as is expected, the marginal belief functional is the expectation with respect to the marginal probability $marg_J(P)$. □

4 Main Result

In this section, we present our main characterization result. We first define the behavioral property Multi-Issue Uncertainty Aversion (MIUA) in Subsection 4.1. Then, using the concept of a core that we introduce in Subsection 4.2, we characterize the set of belief functionals consistent with MIUA (Theorem 1). We also discuss a related behavioral property, Local Preference for Hedging (LPH), and its characterization (Theorem 2).

4.1 Multi-Issue Uncertainty Aversion

We begin this subsection by stating the Uncertainty Aversion axiom of GS.[10]

Definition 3 (Uncertainty Aversion)**.** The preference relation $\succsim$ is *uncertainty averse* if for any acts $f, g \in \mathcal{F}$ and $\alpha \in [0,1]$, $f \sim g$ implies $\alpha f + (1-\alpha) g \succsim f$.

Uncertainty Aversion requires that a mixture of two indifferent acts be weakly preferred to each of the two acts.[11] This property is motivated by the hedging effect of the mixture. The effect can be clearly seen when we consider two acts f and g such that act f delivers

[10]This axiom is introduced by Schmeidler (1989).

[11]If $\succsim$ is an invariant biseparable preference, which is the case in GS, Uncertainty Aversion implies that a mixture of finitely many (possibly more than two) pairwise indifferent acts is weakly preferred to each of them.

preferable lotteries on some event, say E, and the other act g does so on the complement of E. In this case, a mixture of f and g smooths out the variations involved in the acts, which helps the DM avoid uncertainty about the future events.

In our multi-issue environment, however, a mixture of acts may also change the set of relevant issues. Formally, if J_f, J_g, and $J_{\alpha f+(1-\alpha)g}$ are the sets of issues relevant to f, g, and their mixture $\alpha f + (1-\alpha)g$, respectively, then

$$J_f \triangle J_g \subset J_{\alpha f+(1-\alpha)g} \subset J_f \cup J_g \,.\text{[12]} \tag{5}$$

When f is mixed with g to make the mixture $\alpha f+(1-\alpha)g$, some issues in $J_f \cap J_g$ may become irrelevant since the variations in g regarding those issues may exactly offset the variations in f. On the other hand, the issues in $J_f \backslash J_g$ remain relevant and, moreover, those in $J_g \backslash J_f$ are newly added to the set of issues that determine the final lottery obtained. Thus, the mixture weakly enlarges the set of relevant issues if $J_f \cap J_g$ is empty and weakly shrinks it if $J_g \backslash J_f$ is empty. The change is indecisive otherwise. These changes being considered, Uncertainty Aversion loses its appeal in this multi-issue environment.

The property MIUA mainly describes the DM's behavior when a mixture expands the set of relevant issues. In light of the observation above, a mixture unambiguously increases the number of relevant issues when acts depend on separate issues with no intersection. MIUA requires that a negative effect is generated in that case.

Definition 4 (Multi-Issue Uncertainty Aversion)**.** The preference relation $\succsim$ exhibits *Multi-Issue Uncertainty Aversion (MIUA)* if the following holds: For any distinct issues $i_1, \cdots, i_m \in \mathcal{I}$, any acts $f_1 \in \mathcal{F}_{i_1}, \cdots, f_m \in \mathcal{F}_{i_m}$, $g_1, \cdots, g_n \in \mathcal{F}$, and any weights $\alpha_1, \cdots, \alpha_m; \beta_1, \cdots, \beta_n \in [0,1]$ with $\sum_{k=1}^m \alpha_k = \sum_{l=1}^n \beta_l = 1$, if $f_k \sim f_{k'}$ and $g_l \sim g_{l'}$ for all k, k', l, l', then

$$\alpha_1 f_1 + \cdots + \alpha_m f_m = \beta_1 g_1 + \cdots + \beta_n g_n \quad \text{implies} \quad f_1 \succsim g_1 \,.$$

We provide two interpretations of MIUA. For convenience, let

$$h = \alpha_1 f_1 + \cdots + \alpha_m f_m = \beta_1 g_1 + \cdots + \beta_n g_n \,. \tag{6}$$

The first interpretation is given by comparing the utility difference between f_1 and h with that between g_1 and h (See Figure 2).[13] MIUA requires that f_1 be preferred to g_1, which means that before the mixtures of f_k's and g_l's are taken, f_k has a higher utility level than

[12] $J_f \triangle J_g$ denotes the symmetric difference of them.

[13] In Figure 2, the utility level of f_1 is above and that of g_1 is below the utility level of the mixture. While the utility level of f_1 is required by MIUA to be higher than that of the mixture, the utility level from g_1 is not necessarily lower than that from the mixture.

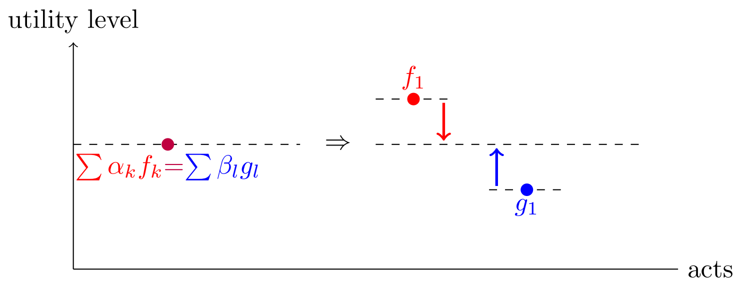

Figure 2: The first interpretation of Multi-Issue Uncertainty Aversion

g_l for all k and l. Through mixtures, however, both groups of acts reach the same act h and, in particular, the same utility level. This means that the change in the utility level from the mixture of f_k's is more negative than that from the mixture of g_l's. Since f_k's do not share any relevant issues, the former mixture only expands the set of relevant issues with no variation smoothed out, i.e., no hedging. On the other hand, hedging is not ruled out in the latter mixture. Under MIUA, the former type of mixture is always less preferred than the other. This can be interpreted as the DM's aversion to increase in uncertain dimensions.

The statement of MIUA does not rule out the possibility that each mixture in (6) is actually a single act (when $\alpha_1 = 1$ or $\beta_1 = 1$). Such cases warrant specific interpretations. If $\alpha_1 = 1$, MIUA implies that the mixture of g_l's is preferred to each g_l. This is because of a potential hedging effect between the acts g_l's. In particular, when g_l's depend on the same issue, MIUA requires exactly what Uncertainty Aversion of GS assumes. In contrast, if $\beta_1 = 1$, MIUA implies that the mixture of the acts f_k's is less preferred to each f_k. This means that the multi-issue consideration resulting from the mixture of f_k's indeed generates a negative effect.

The alternative interpretation comes from the opposite of mixture, or division of a mixed act (See Figure 3). Suppose that there is a group of N agents who share the same preference $\succsim$, and that they want to equitably divide the act h (the mixture). By equitable division, we mean that each agent is indifferent between her own and any other agent's portion resulting from the division. For heuristic purposes, we assume here that an act can be divided into fractions. Then, one obvious way of fairly dividing h is to give each agent $1/N$ 'unit' of h since $h = 1/N \cdot h + \cdots + 1/N \cdot h$.[14] Alternatively, given that $h = \sum_{k=1}^{m} \alpha_k f_k$, another way is to split the agents into m groups of $\alpha_1 \cdot N, \cdots, \alpha_m \cdot N$ agents and give $1/N$ unit of f_1 to each agent in the first group, $1/N$ unit of f_2 to each agent in the second group, and so on.

[14]This division of an act must be done probabilistically. For example, one can draw a random number among 1 through N and then give (undivided) act h to the corresponding agent. One can think of similar probabilistic ways for other divisions, too. We believe that the interpretation we provide is still valid even with this more rigorous way of division.

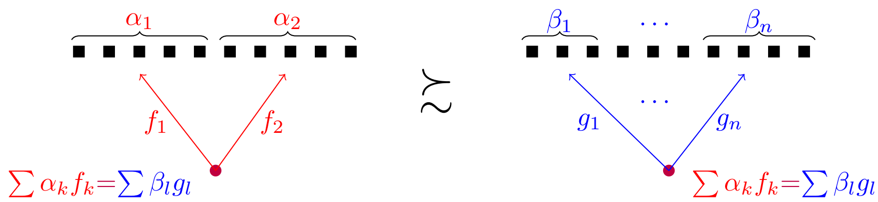

Figure 3: The second interpretation of Multi-Issue Uncertainty Aversion: The squares represent agents. So, there are $N = 10$ agents in this figure. On the left-hand side of this figure, each of the first $\alpha_1 \cdot N = 5$ agents receive $1/10$ unit of f_1, and the rest $\alpha_2 \cdot N = 5$ agents receive $1/10$ unit of f_2. The other side of the figure should be similarly read. The agents prefer the left-hand-side division under MIUA.

Still another way is to divide h using the equation $h = \sum_{l=1}^{n} \beta_n g_n$ in a similar way. MIUA says that among all these ways of division, the agents prefer the division into $f_1, \cdots, f_m$ the most. Since f_k's depend only on a single issue while g_l's do not necessarily, this preference implies that those agents want to avoid multi-issue uncertainty.

MIUA has threefold implications. First, it imposes Uncertainty Aversion of GS among i-acts for each $i \in \mathcal{I}$. Given any i-acts $g_1, g_2 \in \mathcal{F}_i$, if we take $m = 1$, $i_1 = i$, $f_1 = \beta_1 g_1 + \beta_2 g_2$, and $\alpha_1 = 1$ in Definition 4, then MIUA implies that the mixture of g_1 and g_2 is weakly preferred to each of them. Second, MIUA requires that the DM prefer a mixture if it reduces the number of relevant issues to one or zero (in which case the mixture is a constant act). This can be seen by taking $\alpha_1 = 1$ in Definition 4. It means that the DM is less uncertainty averse about each individual issue than about a set of multiple issues.[15] Third, MIUA implies that a mixture of indifferent acts that depend on distinct issues is weakly less preferred to each of those individual acts, as we have discussed. This is consistent with the ranking (1) we suggested in our thought experiment in Section 1.[16]

4.2 Core

We proceed to discuss the implication of MIUA for utility functions. We begin by defining the core and marginal core of a belief functional.

[15]This second implication is related to the Local Preference for Hedging defined in a later subsection.

[16]We point out here that MIUA can be written in a more general way: We can have each act f_k depend on a set of issues, say J_k, instead of a single issue i_k. In that case, the sets J_k's must be pairwise disjoint so that the mixture of f_k's certainly increases the uncertain dimensions. In terms of interpretation, this alternative definition has no difference with Definition 4 except that the DM regards each group of issues in J_k as a single issue. Moreover, our characterization in Subsection 4.3 can be easily generalized, too. Since there is no conceptual difference, we keep our simpler version of MIUA throughout this paper.

Definition 5. Let $\mathbb{B} : \mathbb{R}^S \to \mathbb{R}$ be a belief functional. The *core* of $\mathbb{B}$ is defined by

$$core(\mathbb{B}) = \{P \in \Delta(S) : P \cdot v \geq \mathbb{B}(v),\ \forall v \in \mathbb{R}^S\}.$$

The *marginal core* of $\mathbb{B}$ with respect to a nonempty subset $J \subset \mathcal{I}$ is defined by

$$mc_J(\mathbb{B}) = \{p \in \Delta(S_J) : p \cdot v \geq \mathbb{B}_J(v),\ \forall v \in \mathbb{R}^{S_J}\}.$$

The *core (marginal core, resp.)* of an invariant biseparable preference is the core (marginal core, resp.) of its representing belief functional.

For any fixed utility index u, each probability on S induces an SEU preference over $\mathcal{F}$. By definition, a probability P in the core of $\mathbb{B}$ satisfies $P \cdot u(f) \geq \mathbb{B}(u(f))$ for any act $f \in \mathcal{F}$. Moreover, for each lottery $x \in \mathcal{L}$, $P \cdot u(\bar{x}) = u(x) = \mathbb{B}(u(\bar{x}))$.[17] Thus, for any act $f \in \mathcal{F}$ and any lottery $x \in \mathcal{L}$, $P \cdot u(\bar{x}) \geq P \cdot u(f)$ implies $\mathbb{B}(u(\bar{x})) \geq P \cdot u(f) \geq \mathbb{B}(u(f))$. That is, whenever a constant act $\bar{x}$ is preferred to another act f, which possibly involves uncertainty, under the preference induced by P (denoted by $\succsim_P$), the same is true under the one induced by $\mathbb{B}$ (denoted by $\succsim_{\mathbb{B}}$).[18] So, $core(\mathbb{B})$ is the set of all probabilities that induce an SEU preference that is more willing to take on uncertainty than $\succsim_{\mathbb{B}}$. In light of this, we view the core as a measure of aversion to uncertainty. A larger core in terms of set inclusion is associated with a higher degree of uncertainty aversion.[19]

The marginal core $mc_J(\mathbb{B})$ is similarly defined as the core. It can be alternatively viewed as the core of the marginal preference over $\mathcal{F}_J$. Hence, we view the marginal core as a measure of aversion to uncertainty about issues in J.

Example 3. *As we saw in Example 1, the belief functionals of SEU, CEU, and MEU are given by* $\mathbb{B}(v) = P \cdot v$, $\mathbb{B}(v) = \int_S v\, d\nu$, *and* $\mathbb{B}(v) = \min_{P \in C} P \cdot u(f)$, *respectively, for some probability* $P \in \Delta(S)$, *some capacity* ν *on* S, *and some nonempty closed convex set* $C \subset \Delta(S)$.

(1) For SEU, $core(\mathbb{B}) = \{P\}$ *and* $mc_J(\mathbb{B}) = \{marg_J(P)\}$ *for each nonempty* $J \subset \mathcal{I}$.

[17] Constant linearity of $\mathbb{B}$ implies that it is *normalized*: For any $c \in \mathbb{R}$, $\mathbb{B}(\bar{c}) = c$. It can be seen that $\mathbb{B}(\bar{0}) = 0$ and $\mathbb{B}(\bar{c}) = \mathbb{B}(\bar{0} + \bar{c}) = 0 + c = c$ from constant linearity.

[18] In the terminology of Ghirardato and Marinacci (2002), $\succsim_P$ is less ambiguity averse than $\succsim_{\mathbb{B}}$. See Definition 9 in Section 5.

[19] Ghirardato and Marinacci (2002) show that a biseparable preference is more ambiguity averse than another biseparable preference only if the core of the former includes that of the latter (Proposition 16). If they are MEU preferences, then the two conditions are equivalent (Theorem 17).

(2) For CEU, core($\mathbb{B}$) *coincides with the core of the capacity of* ν*. That is,*

$$core(\mathbb{B}) = \{P \in \Delta(S) : P(E) \geq \nu(E), \forall E \subset S\}.$$

By an abuse of notation, we will also denote this set by core(ν) *whenever it is convenient. The marginal core is the core of its marginal capacity, which can be written as, for each nonempty* $J \subset \mathcal{I}$*,*

$$mc_J(\mathbb{B}) = \{marg_J(P) \in \Delta(S_J) : P(E) \geq \nu(E), \forall E \in \mathcal{A}_J\}.$$

(3) For MEU, $core(\mathbb{B}) = C$ *and* $mc_J(\mathbb{B}) = \{marg_J(P) \in \Delta(S_J) : P \in C\}$ *for each nonempty* $J \subset \mathcal{I}$*.*[20]

□

Our characterization of MIUA in the following subsection is achieved by comparing the core and marginal cores of the DM's preference. Even though the elements in them have different dimensions, we can compare the sets by considering appropriate marginals of probabilities in the core. The following lemma shows that there is a set inclusion relationship between those sets.

Lemma 1. *Let* $\mathbb{B} : \mathbb{R}^S \to \mathbb{R}$ *be a belief functional. Then, for each nonempty subset* $J \subset \mathcal{I}$*,*

$$\left\{marg_J(P) \in \Delta(S_J) : P \in core(\mathbb{B})\right\} \subset mc_J(\mathbb{B}).$$

Now we turn to define the condition called *exhaustiveness* of a core. The condition is meant to capture the DM's high degree of aversion to uncertainty about the entire set of issues collectively *relative to* uncertainty about individual issues separately. As we argued above, we interpret a larger core as a higher degree of aversion to uncertainty. So, exhaustiveness requires that the core of a preference be sufficiently large. From Lemma 1, we know that the set of marginals of core probabilities is bounded above by $mc_J(\mathbb{B})$ in terms of set inclusion. Exhaustiveness holds if the upper bound is well achieved for each issue i.

Definition 6. Let $\mathbb{B} : \mathbb{R}^S \to \mathbb{R}$ be a belief functional. Then, the core of $\mathbb{B}$ is *exhaustive* if

$$\left\{(marg_i(P))_{i\in\mathcal{I}} \in \times_{i\in\mathcal{I}}\Delta(S_i) : P \in core(\mathbb{B})\right\} = \times_{i\in\mathcal{I}} mc_i(\mathbb{B}). \tag{7}$$

[20]For SEU and MEU, $mc_J(\mathbb{B}) = \{marg_J(P) \in \Delta(S_J) : P \in core(\mathbb{B})\}$. However, this is not true for CEU. For example, suppose $S_1 = \{s_1, t_1\}$, $S_2 = \{s_2, t_2\}$, and the capacity satisfies $\nu(E) = 0.5$ for every nonempty proper subset E of $S_1 \times S_2$. Then, $core(\mathbb{B})$ is empty. However, $mc_1(\mathbb{B})$ is not empty. In fact, it is a singleton that contains the uniform probability on s_1 and t_1.

Since Lemma 1 implies that the set on the right-hand side of (7) includes that on the left-hand side, only the opposite direction of inclusion matters. So, we can restate Definition 6 as follows: The core is exhaustive if for any tuple $(p_i)_{i\in\mathcal{I}} \in \times_{i\in\mathcal{I}} mc_i(\mathbb{B})$, there exists a core probability whose marginal on S_i is equal to p_i for all $i \in \mathcal{I}$. This condition is tightly connected to our behavioral property MIUA as we show in the following subsection.

4.3 Characterization of MIUA

We now present the characterization of MIUA, which is our main result. A marginal belief functional $\mathbb{B}_J$ is *superadditive* if $\mathbb{B}_J(v + w) \geq \mathbb{B}_J(v) + \mathbb{B}_J(w)$ for all $v, w \in \mathbb{R}^{S_J}$.

Theorem 1. *Suppose $(u, \mathbb{B})$ is an invariant biseparable representation of the preference relation $\succsim$. Then, the following are equivalent:*

(1) The preference relation $\succsim$ exhibits Multi-Issue Uncertainty Aversion.

(2) The core of $\mathbb{B}$ is exhaustive, and the marginal belief functional $\mathbb{B}_i$ is superadditive for each $i \in \mathcal{I}$.

In Subsection 4.1, we discussed the threefold implications of MIUA. The first was that it imposes Uncertainty Aversion of GS on each $\mathcal{F}_i$. This is equivalent to the superadditivity of $\mathbb{B}_i$, or the DM being an MEU maximizer when considering i-acts only.[21] The second was that the DM prefers a mixture of pairwise indifferent acts if the set of relevant issues collapses to a singleton or the empty set. This implies, according to the result that will be presented in Subsection 4.4,

$$\left\{ marg_i(P) \in \Delta(S_i) : P \in core(\mathbb{B}) \right\} = mc_i(\mathbb{B}), \quad \forall i \in \mathcal{I}. \tag{8}$$

In view of Lemma 1, equation (8) means that the core of $\mathbb{B}$ is sufficiently large relative to each marginal core $mc_i(\mathbb{B})$. This already suggests that the DM is highly averse to uncertainty about the entire set of issues relative to individual issues, yet equation (8) is weaker than what exhaustiveness requires. The exhaustiveness requires $core(\mathbb{B})$ to be even larger, so that the marginal cores are *jointly* covered by $core(\mathbb{B})$. This additional requirement comes from the full strength of MIUA including the third implication which says that a mixture of acts depending on distinct issues becomes less desirable.

Theorem 1 achieves the characterization of utility functions through comparing a core and marginal cores. The concept of a core has been used to compare two different decision makers' aversion to uncertainty, for example, in Ghirardato and Marinacci (2002). Our

[21] See Lemma B.8 in Appendix B for details.

result shows that it can also be used to compare a single decision maker's different degrees of aversion to uncertainty regarding different sets of acts. Or, we can at least say that MIUA behaviorally describes the kind of uncertainty aversion that is captured by the relative sizes of cores and marginal cores.

4.4 Local Preference for Hedging

In this subsection, we introduce another behavioral property that can be understood as a weakening of MIUA and its characterization in terms of belief functionals. Roughly speaking, under MIUA, the DM prefers a mixture if the set of relevant acts shrinks and dislikes it if the set expands. In some environments or economic models, however, it might be too restrictive to assume both patterns of behavior. The following behavioral property relaxes the second requirement and only demands that the DM prefers a mixture when the set of relevant issues diminishes into a particular set.

Definition 7 (Local Preference for Hedging)**.** Let $J \subset \mathcal{I}$ be a nonempty subset of issues. Preference relation $\succsim$ exhibits *Local Preference for Hedging (LPH) with respect to J* if the following holds: For any pairwise indifferent acts $f_1, \cdots, f_n \in \mathcal{F}$ and any $\alpha_1, \cdots, \alpha_n \in [0,1]$ with $\sum_{k=1}^{n} \alpha_k = 1$, if the mixed act $\alpha_1 f_1 + \cdots + \alpha_n f_n$ belongs to $\mathcal{F}_J$, then

$$\alpha_1 f_1 + \cdots + \alpha_n f_n \succsim f_1 \,.$$

Suppose J is a singleton set, say $\{i\}$. Then, LPH with respect to J is implied by MIUA.[22] In fact, LPH with respect to J has only the first two of the threefold implications of MIUA we discussed. Namely, it imposes Uncertainty Aversion of GS on $\mathcal{F}_i$, and the DM prefers a mixture if it results in an act that depends only on issue i. Even when J includes multiple issues, we can interpret LPH in a similar way. The only difference is that the DM treats all issues in J as a 'big' single issue. In any case, unless J is equal to the whole set $\mathcal{I}$, LPH is strictly weaker than Uncertainty Aversion of GS in that the preference for mixtures is exhibited only *locally* with in $\mathcal{F}_J$.[23]

LPH is also characterized by a sufficiently large core of the preference. However, the notion of largeness should be slightly different from the previously defined exhaustiveness. Instead, we define the J-exhaustiveness of a core as below. Given the definition, it will be easy to see that a core is exhaustive in the sense of Definition 6 if and only if it is $\{i\}$-exhaustive for all issues $i \in \mathcal{I}$.

[22]To see this, take $m = 1$, $i_1 = i$, $f_1 = \alpha_1 g_1 + \cdots + \alpha_n g_n$ in Definition 4.

[23]A plausible preference is the one that exhibits LPH with respect to J for any J with $|J| \leq k$ for some threshold k.

Definition 8. Let $\mathbb{B} : \mathbb{R}^S \to \mathbb{R}$ be a belief functional and J be a nonempty subset of $\mathcal{I}$. Then, the core of $\mathbb{B}$ is *J-exhaustive* if

$$\left\{ marg_J(P) \in \Delta(S_J) : P \in core(\mathbb{B}) \right\} = mc_J(\mathbb{B}) . \tag{9}$$

Again by Lemma 1, an alternative definition of J-exhaustiveness is allowed: For any probability $p_J \in mc_J(\mathbb{B})$, there exists a probability $P \in \Delta(S)$ consistent with p_J in the sense that $marg_J(P) = p_J$. Thus, J-exhaustiveness means that the core of $\mathbb{B}$ is sufficiently large to cover the marginal core $mc_J(\mathbb{B})$. As before, the condition can be interpreted as the DM being greatly averse to uncertainty about the entire set of issues relative to uncertainty about the issues in set J, which we view as an underlying cause of the behavior captured by LPH. We conclude this section by stating the utility function characterization of LPH.

Theorem 2. *Suppose $(u, \mathbb{B})$ is an invariant biseparable representation of the preference relation $\succsim$ and let J be a nonempty subset of $\mathcal{I}$. Then, the following are equivalent:*

(1) The preference relation $\succsim$ exhibits Local Preference for Hedging with respect to J.

(2) The core of $\mathbb{B}$ is J-exhaustive and the marginal belief functional $\mathbb{B}_J$ is superadditive.

5 A Special Case: Additive Marginal Beliefs

In this section, we consider a special case in which the DM has an additive marginal belief on each single issue, but her belief over the entire set of states is not necessarily additive. Such a belief may naturally arise when the DM has a lot of information about the marginal distributions regarding each issue, but is poorly informed about the correlation across issues. The ignorance can be caused by scarcity of simultaneous observations of multiple issues or intractability of data with too many dimensions. In this special case, we can obtain a simpler characterization of MIUA and establish an equivalence between MIUA and other behavioral properties in the literature.

Suppose that $(u, \mathbb{B})$ is an invariant biseparable representation of the preference relation $\succsim$, and that the marginal belief functional $\mathbb{B}_i$ is additive for each $i \in \mathcal{I}$. It can be readily seen that $\mathbb{B}_i$ is additive if and only if $\succsim$ satisfies the Independence axiom on $\mathcal{F}_i$: For all $f, g, h \in \mathcal{F}_i$ and $\alpha \in (0, 1]$, $f \succsim g$ if and only if $\alpha f + (1-\alpha)h \succsim \alpha g + (1-\alpha)h$. This condition can be interpreted as the DM exhibiting no uncertainty aversion about issue i. Then, the DM's uncertainty aversion, if any, only comes from simultaneous consideration of different issues. Let $p_i \in \Delta(S_i)$ be the unique probability that satisfies $\mathbb{B}_i(v) = p_i \cdot v$, for all $v \in \mathbb{R}^{S_i}$, for each $i \in \mathcal{I}$. Then, the marginal core $mc_i(\mathbb{B})$ is the singleton $\{p_i\}$ for all $i \in \mathcal{I}$. So, the exhaustiveness condition of the core becomes simpler as the following lemma states.

Lemma 2. *Suppose that $(u, \mathbb{B})$ is an invariant biseparable representation of the preference relation $\succsim$, and that the marginal belief functional $\mathbb{B}_i$ is additive for all $i \in \mathcal{I}$. Then, the core of $\mathbb{B}$ is exhaustive if and only if it is nonempty.*

If the marginal core $mc_i(\mathbb{B})$ is nonempty for all $i \in \mathcal{I}$, nonemptiness of the core is necessary for exhaustiveness. Lemma 2 says that it is also sufficient if marginal beliefs are additive. This is because if there exists any core probability, by Lemma 1, its marginals will be equal to the unique marginal probabilities in the marginal cores, that is, p_i's.

From this equivalence we see that MIUA is closely related to two other notions of uncertainty aversion in the literature—*Ambiguity Aversion* of Ghirardato and Marinacci (2002) and *Preference for Sure-Diversification* of Chateauneuf and Tallon (2002). We state slightly modified definitions of them below.[24] [25]

Definition 9. The preference relation $\succsim$ exhibits *Ambiguity Aversion* if there exists an SEU preference relation $\succsim'$ such that for all $x \in \mathcal{L}$ and $f \in \mathcal{F}$,

$$\bar{x} \succsim' f \implies \bar{x} \succsim f \quad \text{and} \quad \bar{x} \succ' f \implies \bar{x} \succ f\,.$$

It exhibits *Preference for Sure-Diversification (PSD)* if for any pairwise indifferent acts $f_1, \cdots, f_n \in \mathcal{F}$ and any $\alpha_1, \cdots, \alpha_n \in [0,1]$ with $\sum_{k=1}^{n} \alpha_k = 1$,

$$\alpha_1 f_1 + \cdots + \alpha_n f_n \in \mathcal{F}_{\varnothing} \quad \text{implies} \quad \alpha_1 f_1 + \cdots + \alpha_n f_n \succsim f_1\,.$$

Ambiguity Aversion is an intuitive notion of aversion to uncertainty. In the definition, since $\bar{x}$ is constant, no uncertainty is involved in the act. On the other hand, the act f is potentially an uncertain act whose resulting lotteries depend on the realization of a state. Taking SEU preference relations as the benchmarks with no aversion to uncertainty, Ghirardato and Marinacci (2002) defines Ambiguity Aversion as having a stronger taste for constant acts over uncertain nonconstant acts than some SEU preference.

[24] Ghirardato and Marinacci (2002) define Uncertainty Aversion and Ambiguity Aversion differently. In fact, the definition we provide in Definition 9 is what they call Uncertainty Aversion. However, two notions coincide for invariant biseparable preferences. We use the name Ambiguity Aversion to distinguish it from Uncertainty Aversion of GS.

[25] In the main model of Chateauneuf and Tallon (2002), a choice object is a mapping from states to real numbers, not lotteries. The definition we provide in Definition 9 is a version for the Anscombe-Aumann framework that we use. In fact, they also study Anscombe-Aumann acts and consider a stronger property 'Preference for Sure Expected Utility Diversification' in that case. For invariant biseparable preferences, their stronger property is equivalent to the one we stated in Definition 9.

Preference for Sure-Diversification is a weakening of the Uncertainty Aversion axiom of GS. While the latter requires that a hedging effect be positive in any mixture of pairwise indifferent acts, PSD requires such a mixture to be preferred only when the mixture is a constant act. So, the DM may not prefer a mixture unless it is a complete hedge. Informally, we can regard PSD as LPH with respect to the empty set, so PSD is weaker than our notion of LPH (with respect to any nonempty J), too.

It is known that Ambiguity Aversion is equivalent to nonemptiness of the core for an invariant biseparable preference (Ghirardato and Marinacci, 2002, Theorem 12), and that PSD is equivalent to nonemptiness of the core for a CEU preference (Chateauneuf and Tallon, 2002, Theorem 5). In fact, the latter equivalence holds for an invariant biseparable preference, too. Combining these with Lemma 2, we obtain the following result.

Theorem 3. *Suppose $(u, \mathbb{B})$ is an invariant biseparable representation of the preference relation $\succsim$. Then, Multi-Issue Uncertainty Aversion implies Ambiguity Aversion and Preference for Sure-Diversification. In addition, if $\succsim$ satisfies Independence on $\mathcal{F}_i$ for each $i \in \mathcal{I}$, then the following are equivalent:*

(1) The preference $\succsim$ exhibits Multi-Issue Uncertainty Aversion.

(2) The preference $\succsim$ exhibits Ambiguity Aversion.

(3) The preference $\succsim$ exhibits Preference for Sure-Diversification.

(4) The core of $\mathbb{B}$ is nonempty.

We emphasize three implications deduced from Theorem 3. First, if a decision maker is modeled by an invariant biseparable preference having additive marginal beliefs on each issue and exhibiting Ambiguity Aversion (or PSD), then such an assumption has an implication that the decision maker also exhibits MIUA. Second, MIUA boils down to Ambiguity Aversion and PSD when marginal beliefs are additive. MIUA is meant to capture a decision maker's higher degree of aversion to uncertainty when multiple issues are collectively considered than when each issue is separately considered. Since additive marginal beliefs correspond to zero aversion regarding each issue, MIUA under additive marginal beliefs represents aversion to uncertainty purely about the relationship between issues. The equivalence result in Theorem 3 shows that the imposition of MIUA as a notion of uncertainty aversion in that case is as reasonable as imposing Ambiguity Aversion or PSD. Lastly, MIUA can be regarded as an extension of Ambiguity Aversion from the case in which the relationship between issues is the only source of aversion to uncertainty to the case in which each single issue is also a source of uncertainty aversion. We already have seen that MIUA is an extension of the Uncertainty Aversion axiom of GS from a single-issue to multi-issue environment.

6 Examples

In this section, we analyze utility functions which satisfy the conditions stated in Theorem 1 and hence represent preferences exhibiting MIUA. In Subsection 6.1, we provide two numerical examples of CEU functions with capacities derived by taking the lower envelope of a set of (additive) probabilities. One of them satisfies the exhaustiveness condition, while the other fails. In Subsection 6.2, we discuss the products of MEU preferences and specific CEU preferences (Walley and Fine, 1982; Hendon et al., 1996) that exhibit MIUA.

6.1 CEU preferences with lower-envelope capacities

In this subsection, we put our model in the context of making investments in firms. Suppose there are two firms, denoted by 1 and 2. Whether firm i will default or not is considered as issue i ($\mathcal{I} = \{1, 2\}$). Let $S_i = \{d_i, n_i\}$ for each i. The DM is an investor and there are two possible investment outcomes, good (1) or bad (0). Consider the following three acts, or investment options, N_1, N_2, and D_2 (See also Figure 4):

$$
\begin{aligned}
N_1(d_1d_2) = N_1(d_1n_2) = 0, \quad N_1(n_1d_2) = N_1(n_1n_2) = 1\,;\\
N_2(d_1d_2) = N_2(n_1d_2) = 0, \quad N_2(d_1n_2) = N_2(n_1n_2) = 1\,;\\
D_2(d_1n_2) = D_2(n_1n_2) = 0, \quad D_2(d_1d_2) = D_2(n_1d_2) = 1\,.
\end{aligned}
$$

Under act N_1, the good outcome is realized (for sure) if and only if firm 1 does not default. The act N_2 is similar. So, they are bets on each firm's not defaulting. On the other hand, D_2 is a bet on firm 2's default. Clearly, N_1 is a 1-act while N_2 and D_2 are 2-acts.

Suppose that the DM has a probabilistic belief about each firm's default likelihood, but she is unsure whether the two firms are likely to end up with the same or different results. In particular, she regards any convex combination of the following two probabilities P_1 and P_2 as plausible:

$$
\begin{aligned}
P_1(d_1d_2) = P_1(n_1n_2) = 0.3, \quad P_1(d_1n_2) = P_1(n_1d_2) = 0.2\,;\\
P_2(d_1d_2) = P_2(n_1n_2) = 0.2, \quad P_2(d_1n_2) = P_2(n_1d_2) = 0.3\,.
\end{aligned}
$$

The second row in Figure 4 is presenting P_1 and P_2. Let C denote the set of all convex combinations of P_1 and P_2. Then, consider the lower envelope of C, that is, a set function $\underline{P}$ on S such that for any event $E \subset S$,

$$\underline{P}(E) = \min_{P \in C} P(E)\,.$$

	d_2	n_2
d_1	0	0
n_1	1	1

act N_1

	d_2	n_2
d_1	0	1
n_1	0	1

act N_2

	d_2	n_2
d_1	1	0
n_1	1	0

act D_2

	d_2	n_2
d_1	0.3	0.2
n_1	0.2	0.3

probability P_1

	d_2	n_2
d_1	0.2	0.3
n_1	0.3	0.2

probability P_2

	d_2	n_2
d_1	0.5	0.1
n_1	0.1	0.3

probability Q_1

	d_2	n_2
d_1	0.3	0.1
n_1	0.1	0.5

probability Q_2

Figure 4: The example in Subsection 6.1

By construction, $\underline{P}$ is monotone and hence a capacity. Suppose the DM has a CEU preference represented by the capacity $\underline{P}$ and a utility index with $u(1) = 1$ and $u(0) = 0$. In this case, the core of the DM's belief functional, which is equal to the core of $\underline{P}$, is nonempty. For example, a probability that assigns 0.25 to each state is in the core. Since the marginal belief on each issue is additive, Lemma 2 implies that the core is exhaustive. Thus, from Theorem 1, we know that the DM's preference exhibits MIUA.[26]

Instead of P_1 and P_2, suppose the DM considers the priors that are convex combinations of Q_1 and Q_2 given by

$$Q_1(d_1d_2) = 0.5, \quad Q_1(n_1n_2) = 0.3, \quad Q_1(d_1n_2) = Q_1(n_1d_2) = 0.1\,;$$
$$Q_2(d_1d_2) = 0.3, \quad Q_2(n_1n_2) = 0.5, \quad Q_2(d_1n_2) = Q_2(n_1d_2) = 0.1\,.$$

They are presented at the bottom of Figure 4. In this case, the DM does not entertain a single default probability for each firm. Instead, she thinks the probability is between 0.4 and 0.6 for both firms. However, she thinks that the two firms have the same default probability and that they will end up with the same result with a 0.8 chance. Thus, unlike the previous case, there is not as much uncertainty about the relationship between the two issues as about each issue separately.

Similarly to the previous case, consider CEU with the lower envelope $\underline{Q}$ derived from the

[26]As a test case, consider the acts N_1, N_2, and a mixture $1/2N_1 + 1/2N_2$. Since $\underline{P}(n_1d_2, n_1n_2) = \underline{P}(d_1n_2, n_1n_2) = 0.5$, the DM's utility level from N_1 and N_2 is 0.5. On the other hand, since $\underline{P}(d_1n_2, n_1d_2, n_1n_2) = 0.7$ and $\underline{P}(n_1n_2) = 0.2$, her utility level from the mixtures is $\frac{1}{2} \cdot 0.7 + \frac{1}{2} \cdot 0.2 = 0.45$. Therefore, the DM strictly prefers N_1 and N_2 over their mixture as expected.

convex combinations of Q_1 and Q_2. Then, it can be seen that the core of the CEU belief functional is not exhaustive. For example, even though two marginal probabilities with 0.6 on d_1 and 0.4 on d_2 belong to the marginal cores with respect to issue 1 and 2, respectively, there is no core probability whose marginals are simultaneously consistent with them. So, the DM's preference does not exhibit MIUA.[27]

The examples above illustrate two facts. First, a CEU preference with a lower-envelope capacity can be used to model behaviors exhibiting MIUA, but not always.[28] Second, whether such a preference exhibits MIUA is determined by how permissive the set of priors is in terms of their marginals relative to the correlations across those marginals. In the example above, $\underline{P}$ allows more various correlations and is consistent with MIUA, while $\underline{Q}$ allows more various marginal probabilities and conflicts with MIUA.

6.2 Products of MEU and CEU preferences

It has been discussed in the literature how a decision maker forms an extended belief about multiple issues based on her belief about individual issues. The preference induced by the extended belief can be regarded as a product preference of the decision maker's marginal preferences. In the literature, products of MEU preferences (Gilboa and Schmeidler, 1989) and of specific CEU preferences (Walley and Fine, 1982; Hendon et al., 1996; Ghirardato, 1997) have been studied. We demonstrate in this subsection that some of the product preferences proposed in these papers are consistent with MIUA.

We again assume a binary set of issues: $\mathcal{I} = \{1, 2\}$. Gilboa and Schmeidler (1989) provide the notion of a product preference, discussing an extension of the MEU preference to a two-dimensional set of states. Let u be a utility index function and C_i be a nonempty closed convex subset of $\Delta(S_i)$, for each $i \in \mathcal{I}$. Then, let $\succsim_i'$ be a preference relation on $\mathcal{F}_i$ which is represented by an MEU function

$$\min_{p_i \in C_i} p_i \cdot u(\varphi_i(f)) .$$

Then, define a set of probabilities on S

$$C^G = \overline{co}\big(\{p_1 \times p_2 \in \Delta(S) : p_1 \in C_1, p_2 \in C_2\}\big) , \tag{10}$$

[27] As a counterexample to MIUA, we consider the acts N_1, D_2, and the mixture $1/2N_1 + 1/2D_2$ of them. Since $\underline{Q}(n_1d_2, n_1n_2) = \underline{Q}(d_1d_2, n_1d_2) = 0.4$, the DM's utility levels from acts N_1 and D_2 are both 0.4. On the other hand, since $\underline{Q}(d_1d_2, n_1d_2, n_1n_2) = 0.9$ and $\underline{Q}(n_1d_2) = 0.1$, her utility level from the mixture is $\frac{1}{2} \cdot 0.9 + \frac{1}{2} \cdot 0.1 = 0.5 > 0.4$. Hence, the mixture is strictly preferred as opposed to what MIUA requires.

[28] Nevertheless, if the capacity induces additive marginal beliefs, then such a CEU preference does exhibit MIUA since any probability in the set of priors from which the lower envelope is taken belongs to the core of the preference and hence the core is nonempty.

where $p_1 \times p_2$ is the independent product of p_1 and p_2. The *product* of the two MEU preferences $\succsim_1'$ and $\succsim_2'$ is the MEU preference on $\mathcal{F}$ that is represented by

$$V^G(f) = \min_{P \in C^G} P \cdot u(f)\,.$$

We denote it by $\succsim^G$. By construction of the set C^G, the restrictions of $\succsim^G$ on $\mathcal{F}_1$ and $\mathcal{F}_2$ are $\succsim_1'$ and $\succsim_2'$, respectively. Furthermore, GS proves that the product $\succsim^G$ is the unique MEU extension of $\succsim_1'$ and $\succsim_2'$ that satisfies their (stochastic) independence condition (GS, Proposition 4.2).[29]

This product preference $\succsim^G$ exhibits MIUA. To see this, recall that the core of an MEU belief functional is the set of priors over which the minimum is taken. Hence, C^G is the core and C_1 and C_2 are the marginal cores of $\succsim^G$. The construction of C^G in (10) implies that for any pair of marginal probabilities p_1 and p_2 in the marginal cores, a probability consistent with them, namely $p_1 \times p_2$, exists in the core. Thus, MIUA is guaranteed by Theorem 1. MIUA and Uncertainty Aversion of GS are conflicting properties: The former requires that a mixture of two indifferent 1-act and 2-act be weakly less preferred to each of them, while the mixture must be weakly preferred to both of them under the latter. The product $\succsim^G$ is a boundary case in which such a mixture is precisely indifferent to individual acts. In fact, any invariant biseparable preference that coincides with $\succsim^G$ on each $\mathcal{F}_i$ and evaluates every act uniformly worse in terms of its certainty equivalent exhibits MIUA. This is true because such a preference has a core no smaller than that of $\succsim^G$. The formal statement follows.

Proposition 1. *Assume $\mathcal{I} = \{1, 2\}$. Suppose $(u, \mathbb{B})$ is an invariant biseparable representation of the preference relation $\succsim$, and assume that the marginal preferences $\succsim|_1$ and $\succsim|_2$ have MEU representations. Let V^G be the MEU representation of the product $\succsim^G$ of $\succsim|_1$ and $\succsim|_2$ with the same utility index u. If $\mathbb{B}(u(f)) \leq V^G(f)$ for all $f \in \mathcal{F}$, then the preference relation $\succsim$ exhibits Multi-Issue Uncertainty Aversion.*

Walley and Fine (1982) and Hendon et al. (1996) define products of CEU preferences in a similar way to GS. Their definitions are also suggestive of stochastic independence between issues as GS's.[30] Suppose that $\succsim_1'$ and $\succsim_2'$ are CEU preference relations on $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively, and that their beliefs correspond to capacities ν_1 on S_1 and ν_2 on S_2. Assume

[29] In our terminology, the independence condition holds for an MEU preference in a two-issue case if for any pair of a 1-act and a 2-act, (i) their minimum expected utility levels are commonly achieved by a single probability in the core, and (ii) the corresponding utility acts are stochastically independent to each other as random variables.

[30] Hendon et al. (1996) and Ghirardato (1997) study an independent product of capacities. Whereas the independent product of two marginal probabilities is unique, there is a range of capacities that can be considered as independent products of two marginal capacities. See Proposition 1 of Hendon et al. (1996), for example.

the two capacities are convex so that the belief functionals, i.e., the Choquet integrals, are superadditive.[31] Then, each preference $\succsim_i'$ is also an MEU preference whose corresponding set of priors is $core(\nu_i)$ (Schmeidler, 1986). Given these capacities, consider the following sets of probabilities on the product set S:

$$C^W(\nu) = \{p_1 \times p_2 \in \Delta(S) : p_i \in core(\nu_i)\,,\, \forall i \in \mathcal{I}\}\,;$$
$$C^H(\nu) = \{P \in \Delta(S) : P(E_1 \times E_2) \geq \nu_1(E_1)\nu_2(E_2)\,,\, \forall E_1 \subset S_1,\, \forall E_2 \subset S_2\}\,.$$

The former is studied by Walley and Fine (1982) and the latter by Hendon et al. (1996). The set $C^W(\nu)$ includes all independent products of marginal probabilities in $core(\nu_1)$ and $core(\nu_2)$. It is equal to C^G in (10) without the closed convex hull taken. The set $C^H(\nu)$ includes all probabilities under which each rectangular event is assigned at least as much as some independent product probability would assign to it. We can see that every probability in $C^W(\nu)$ also belongs to $C^H(\nu)$, that is, $C^W(\nu) \subset C^H(\nu)$.

Given these sets, the capacities $\pi^W(\nu)$ and $\pi^H(\nu)$ are defined as the lower envelope of $C^W(\nu)$ and $C^H(\nu)$, respectively:

$$\begin{aligned} \pi^W(\nu)(E) &= \min_{P \in C^W(\nu)} P(E)\,, \quad \forall E \subset S\,; \\ \pi^H(\nu)(E) &= \min_{P \in C^H(\nu)} P(E)\,, \quad \forall E \subset S\,. \end{aligned} \tag{11}$$

If the preference relation $\succsim$ has a CEU representation with one of these capacities, its marginal preferences coincide with $\succsim|_1$ and $\succsim|_2$, so $\succsim$ is indeed an extension of them. We now prove that both CEU preferences represented by capacities in (11) exhibit MIUA. First, because of the set inclusion $C^W(\nu) \subset C^H(\nu)$, it is immediate that $\pi^H(\nu) \leq \pi^W(\nu)$. Moreover, the closed convex hull of $C^W(\nu)$ is equal to C^G in (10) with $C_i = core(\nu_i)$ and $\pi^W(\nu)$ remains the same even if we take the minimum in (11) over C^G instead of $C^W(\nu)$. Thus, the CEU with $\pi^W(\nu)$ is dominated by the MEU function associated with C^G.[32] This implies that for any act f,

$$\int_S u(f) d\pi^H(\nu) \leq \int_S u(f) d\pi^W(\nu) \leq \min_{P \in C^G} P \cdot u(f)\,. \tag{12}$$

We obtain the following corollary of Proposition 1 from the inequalities in (12).

[31] The capacity ν_i is *convex* if $\nu_i(E \cup F) + \nu_i(E \cap F) \geq \nu_i(E) + \nu_i(F)$ for all $E, F \subset S_i$.

[32] Given a nonempty closed convex set C of probabilities on S, it can be seen that for any real-valued function on S, the minimum of the integrals with respect to probabilities in C is greater than or equal to the Choquet integral with respect to the lower envelope of C. See Appendix C, in particular, Fact C.3 for details.

Corollary 1. *Assume $\mathcal{I} = \{1, 2\}$, and for each $i \in \mathcal{I}$, let ν_i be a convex capacity on S_i. If the preference relation $\succsim$ is a CEU preference represented by a capacity $\pi^W(\nu)$ or $\pi^H(\nu)$, then it exhibits Multi-Issue Uncertainty Aversion.*

Even though we have only considered the case with binary issues as the original papers do, the three product preferences above can be easily extended to cases with more than two issues. Moreover, all of the extended versions can still be proved to exhibit MIUA in the same way we have done in this subsection.

7 Discussion

7.1 Implication for under-diversification

Many studies in finance propose explanations for under-diversification in asset markets, connecting it to uncertainty aversion. Some of them study models with investors who have uncertainty about parameters of asset return distributions, their mean and variance, and show that limited-participation may occur: The investors decide not to hold any position in some assets available in the market (Dow and Werlang, 1992; Cao et al., 2005; Easley and O'Hara, 2009; Boyle et al., 2012). These studies assume the MEU preference of GS and argue that a large set of parameters, which means a large amount of uncertainty, may cause non-participation. Boyle et al. (2012) suggest that such uncertainty comes from less familiarity with assets. For example, investors may be more familiar with—and have more accurate estimates of—relevant parameters about home-country stocks than foreign ones, leading them to invest only in the home-country market.

Others focus on the correlation between returns of assets rather than the moments of individual asset return distributions (Jiang and Tian, 2016; Huang et al., 2017; Liu and Zeng, 2017). These papers are related to ours in that they attribute anti-diversifying behavior to uncertainty about the relationship between issues—or, between asset returns in their papers.

However, all of these papers use the MEU preference to describe investors' uncertainty aversion. Under MEU, a mixture of indifferent acts must be weakly preferred to each of them. It means that if an investor in their models holds only a subset of available assets, it is not because the investor dislikes betting on multiple assets simultaneously, but because the assets outside the set are relatively undesirable. In other words, for under- or anti-diversification to arise in their models, some conditions regarding asset returns or prices need to be met so that some assets become unattractive. For example, in Boyle et al. (2012), investors do not invest in a foreign stock when they are assumed to have poor estimates about its returns.

Instead, we regard under-diversification as a matter of preference, which may arise even when each asset is considered equally desirable. We have proposed a way to directly model an investor who does not diversify her portfolio, avoiding uncertainty about multiple issues. This may help address the issue Gorton and Metrick (2013) pointed out in the following quote:

> *There are some features of securitization that seem important but are not directly addressed by the theoretical literature to date. One issue is the fact that the creation of asset-backed securities always involves pooling loans that are homogeneous, that is, a pool consists exclusively of auto receivables, or credit-card receivables. It is not the case that different asset classes are mixed, even when the originator in fact originates many different asset classes. The theories suggest that diversification of the loan pool is important, but we do not observe that in the world. Asset classes are sold separately.* (Gorton and Metrick, 2013, p.42)

In our terminology, auto loans and credit-card loans are different issues. It may be complicated to discover how they are related in terms of default possibility. So, investors may find any securities simultaneously involving them unattractive, which might explain why no such securities exist. MIUA provides a way to model that kind of preference. With the property assumed, investors do not look for well-diversified securities because they do not want them, not because they have a strong preference for a particular class of loans.

We conclude this subsection by stating a proposition that has an implication for the lack of securitization documented above. It says that in our basic framework, MIUA implies that the average value of some acts that depend on distinct issues is higher than the value of the 'average act.'

Proposition 2. *Suppose the preference relation $\succsim$ is an invariant biseparable preference and exhibits Multi-Issue Uncertainty Aversion.*[33] *For any distinct issues $i_1, \cdots, i_n \in \mathcal{I}$, any acts $f_1 \in \mathcal{F}_{i_1}, \cdots, f_n \in \mathcal{F}_{i_n}$, any lotteries $x_1, \cdots, x_n \in \mathcal{L}$, and any weights $\alpha_1, \cdots, \alpha_n \in [0,1]$ with $\sum_{k=1}^n \alpha_k = 1$, if $f_k \sim \bar{x}_k$ for all k, then*

$$\textstyle\sum_{k=1}^n \alpha_k \bar{x}_k \succsim \sum_{k=1}^n \alpha_k f_k \,. \tag{13}$$

To better understand Proposition 2 in the context of securities, suppose that each outcome in $\mathcal{Z}$ is a monetary value ($\mathcal{Z} = \mathbb{R}$) and that the DM's utility index u is linear on $\mathbb{R}$.

[33] Assuming an invariant biseparable preference is not necessary for the statement of this proposition. It is sufficient to assume completeness, transitivity, and Certainty Independence (Axiom 2) that is stated in Appendix A.

	r_2	b_2
r_1	1	1
b_1	0	0

R_1

	r_2	b_2
r_1	1	0
b_1	0	1

Same

	r_2	b_2
r_1	0	0
b_1	1	1

B_1

	r_2	b_2
r_1	0	1
b_1	1	0

Diff

Figure 5: These four bets are used in the experiment of Epstein and Halevy. The number 1 represents a monetary prize (for sure) and 0 no prize (for sure). For example, choosing the act R_1, a subject receives a monetary prize if the ball from the first urn is red, and no prize if it is black.

Then, each lottery x_k in Proposition 2 can be selected to be a degenerate lottery that gives a monetary value, say W_k. If we consider f_k as a security, $f_k \sim \bar{x}_k$ means that W_k is the DM's willingness to pay (WTP) for the security. Then, (13) implies that the DM's WTP for the mixed security $\sum_{k=1}^{n} \alpha_k f_k$ is lower than the average WTP $\sum_{k=1}^{n} \alpha_k W_k$.[34] Consequently, when various securities are sold to a group of investors exhibiting MIUA, it generates a higher revenue to sell them individually than to mix them all and sell the same mixed security to every investor. This is closely related to our second interpretation of MIUA in Subsection 4.1: A group of decision makers sharing the same preference exhibiting MIUA can be better off by splitting a mixed act into multiple acts that each of which depends on a single issue.

7.2 Experiment of Epstein and Halevy (2019)

Epstein and Halevy (2019) conduct a laboratory experiment similar to our thought experiment introduced in Section 1. In their experiment, each ball is only colored (red or black) without a letter, but instead, there are two urns from each of which one ball is drawn. So, we can think of a two-by-two set of states as we did in our thought experiment, but with each issue being the ball's color drawn from each urn. The subjects in their experiment compare two sets of bets depicted in Figure 5. Bets R_1 and B_1 are bets on the ball's color from the first urn. On the other hand, bets *Same* and *Diff* are bets on the two balls having the same and different colors, respectively.

[34] This is true more generally when u is concave on $\mathbb{R}$ since in that case, the WTP for $\sum_{k=1}^{n} \alpha_k \bar{x}_k$ is lower than $\sum_{k=1}^{n} \alpha_k W_k$. That is, $\sum_{k=1}^{n} \alpha_k W_k \geq$ (WTP for $\sum_{k=1}^{n} \alpha_k \bar{x}_k$) $\geq$ (WTP for $\sum_{k=1}^{n} \alpha_k f_k$).

They find that subjects display rankings

$$R_1 \succsim \mathit{Same} \text{ and } B_1 \succsim \mathit{Diff} \tag{14}$$

with at least one of them strict. They say that this preference is plausible "if there is *greater aversion* to ambiguity about the relation between urns than to ambiguity about the bias of (the first urn and hence, presumably) any single urn." (Epstein and Halevy, 2019, p.671) This idea is very closely related to what we have discussed in this paper. However, our behavioral property MIUA cannot say much about the preference in (14). This is because the bets Same and Diff depend on two different issues, but they are not a mixture of two options that depend on colors from different urns.

Instead, we propose the behavioral property below, stated for the binary-issue case, to properly capture the preference in (14). As an additional notation, for acts $f_1, \cdots, f_{n+1} \in \mathcal{F}$ and pairwise disjoint events $E_1, \cdots, E_n \subset S$, $f_1 E_1 \cdots f_n E_n f_{n+1} \in \mathcal{F}$ denotes the composite act that coincides with f_k on E_k for each $k = 1, \cdots, n$, and with f_{n+1} on $S \backslash (E_1 \cup \cdots \cup E_n)$.

Property EH: *For any distinct i, j in $\mathcal{I} = \{1, 2\}$, for any i-acts $f_1, \cdots, f_{n+1} \in \mathcal{F}_i$, and for any pairwise disjoint j-events $E_1, \cdots, E_n \in \mathcal{A}_j$,*

$$f_1 \sim \cdots \sim f_{n+1} \quad \textit{implies} \quad f_1 \succsim f_1 E_1 \cdots f_n E_n f_{n+1} \,.$$

In the experiment above, if we take $f_1 = R_1$, $f_2 = B_1$, and $E_1 = \{r_1 r_2, b_1 r_2\}$, then $f_1 E_1 f_2 = \mathit{Same}$. Similarly, if we take $f_1 = B_1$ and $f_2 = R_1$, then $f_1 E_1 f_2 = \mathit{Diff}$. Thus, if bets R_1 and B_1 are indifferent, Property EH implies (14) except that one of the two relations must be strict.

The interpretation of Property EH is similar to that of MIUA. If the DM is an SEU maximizer—not concerned about the number of relevant issues—and $f_1, \cdots, f_n$ are pairwise indifferent, then f_1 must be indifferent to the composite act $f_1 E_1 \cdots f_n E_n f_{n+1}$. However, if she additionally considers an expansion of the set of relevant issues as undesirable, she will prefer an i-act f_1 to the composite act $f_1 E_1 \cdots f_n E_n f_{n+1}$ that possibly depends on issues 1 and 2 simultaneously.

Whereas we have focused on a mixture of acts that depend on distinct issues as a way to make many issues entangled, Epstein and Halevy's experiment shows that there is another way to do so, which is to take a composition of acts that depend on the same issue. While we do not have a full utility characterization of Property EH, we can show that the capacity $\pi^H(\nu)$ defined by (11) in Subsection 6.2, is consistent with Property EH in addition to MIUA.

Proposition 3. *Assume $\mathcal{I} = \{1, 2\}$ and let ν_i be a convex capacity on S_i for each $i \in \mathcal{I}$. If $\succsim$ is a CEU preference with capacity $\pi^H(\nu)$, then it satisfies Property EH.*

The key property of $\pi^H(\nu)$ that we use to prove Proposition 3 is that it has a sufficiently large core relative to the marginal cores $core(\nu_1)$ and $core(\nu_2)$: For any tuple $(p_{s_j})_{s_j \in S_j}$ with $p_{s_j} \in core(\nu_i)$, there exists $P \in core(\pi^H(\nu))$ under which the conditional probability on S_i given any $s_j \in S_j$ is p_{s_j} $(i \neq j)$. This is reminiscent of the exhaustiveness that characterizes MIUA. Thus, we expect that a similar characterization using a core and marginal cores as in Theorem 1 will also be possible regarding Property EH even though MIUA and Property EH are behaviorally independent of each other.

7.3 Experiment of Ellsberg (1961)

In this subsection, we briefly discuss how our idea is related to the seminal work of Ellsberg (1961). One of the thought experiments he proposes is as follows. An urn contains 100 balls that are either red or black, and one ball is to be drawn. A subject may choose to bet on the ball's color to receive \$100 if the color is matched. Alternatively, she may engage in a lottery that dispenses \$100 with a half chance. These options are depicted below.

red	black
1	0

bet on red

red	black
0	1

bet on black

red	black
1/2	1/2

lottery

The lottery can be regarded as a mixture of the two bets on colors. As documented in the literature, many people tend to choose the lottery rather than the other bets when the numbers of red and black balls are unknown. The lottery is attractive because the uncertain issue about the colors is completely removed by mixing. GS's Uncertainty Aversion axiom is predicated on this observation.

In comparison with Ellsberg (1961), our paper is about the opposite force of a mixture. In other words, we have studied choices when a mixture introduces a new issue rather than removing an existing one. Although mixing always goes hand in hand with removing an issue in a single-issue environment such as Ellsberg's experiment, it may work in the opposite way in our multi-issue environment.

7.4 Non-product set of states

In our model, we use the product set S to represent a multi-issue environment. Although the product structure is natural and widely applicable, we believe that we may, if necessary, dispense with it in performing our analysis. More important are the algebras of i-events based on which i-acts are defined. In fact, we can start with an arbitrary finite set of states and some algebras on it that represent issues. To be more concrete, let S be a finite set of states which is not necessarily a product set, and suppose that for each issue $i \in \mathcal{I}$, $\mathcal{A}_i$ is an algebra on it.[35] Then, for a nonempty subset of $J \subset \mathcal{I}$, we can say that an event is a J-event if it belongs to the algebra, denoted by $\mathcal{A}_J$, that is generated by the union $\bigcup_{i \in J} \mathcal{A}_i$ of individual algebras. An act is a J-act if it is measurable with respect to $\mathcal{A}_J$. A marginal belief functional $\mathbb{B}_J$ can be defined as a real-valued function on the partition of S that generates $\mathcal{A}_J$. Likewise, other terminology we defined in our model can be easily modified. The results analogous to the ones we presented in this paper are believed to hold under this new setting, too, as we do not see any place in our analysis where this alternative approach would fail.

We should be cautious, however, about what collection of algebras is acceptable as a representation of issues. One of the merits of having a product structure is that two algebras, say $\mathcal{A}_J$ and $\mathcal{A}_{J'}$, have only the trivial intersection $\{\varnothing, S\}$ if J and J' are disjoint. So, a mixture of a J-act and J'-act obviously increases the set of relevant issues, making a $(J \cup J')$-act. This is not necessarily true in the alternative approach. For instance, suppose that $S = \{s_1, s_2, s_3\}$, $\mathcal{I} = \{a, b, c\}$ and algebras $\mathcal{A}_a$, $\mathcal{A}_b$, and $\mathcal{A}_c$ are generated by the partitions $\{\{s_1\}, \{s_2, s_3\}\}$, $\{\{s_2\}, \{s_3, s_1\}\}$, and $\{\{s_3\}, \{s_1, s_2\}\}$, respectively. Then, for two distinct lotteries $x, y \in \mathcal{L}$, an act $f_a = \bar{x}\{s_1\}\bar{y}$ is an a-act and another act $f_b = \bar{x}\{s_2\}\bar{y}$ is a b-act. However, the half-and-half mixture of them is

$$\frac{1}{2} f_a + \frac{1}{2} f_b = \left(\frac{1}{2}\bar{x} + \frac{1}{2}\bar{y}\right) \{s_1, s_2\}\, \bar{y}\,,$$

which is a c-act. Thus, the mixture is not necessarily undesirable if the DM is well informed about issue c, even if she is unsure about the relationship between issues a and b. Our property MIUA is less appealing for this particular structure in which the issues turn out to overlap.

Generalizing the concept of issues as we have demonstrated above may be useful in analyzing preferences that are not based on a product state space. For example, it might be of interest to see how an investor makes a choice between options that depend on a firm's future profits. In this case, the natural set of states to use is the real line, which is not a product of multiple sets. However, the investor may well implicitly consider several issues about the firm—competition with other firms, new technologies, and so on—that impact the

[35]Recall that in our model, the algebra $\mathcal{A}_i$ is induced by the product structure of S. Here, we consider $\mathcal{A}_i$ given as a primitive instead.

firm's profit. Such issues can still be incorporated into our model if we take the alternative way described above.

8 Conclusion

We studied a decision problem under uncertainty about multiple issues. We constructed a multi-issue environment by explicitly imposing a product structure on the set of states in the Anscombe-Aumann framework. In this environment, we could observe that certain mixtures of pairwise indifferent acts may increase the number of relevant issues, thus making a more uncertain alternative. This motivated a new pattern of uncertainty averse behavior that conflicts with the prominent notion of Uncertainty Aversion of Gilboa and Schmeidler (1989). We provided a behavioral property, Multi-Issue Uncertainty Aversion, that captures a decision maker's aversion to alternatives that simultaneously depend on many issues and hence are highly uncertain. Then, we focused on the class of invariant biseparable preferences to see what conditions on a belief functional are consistent with the new behavioral property. We showed in our main result that exhaustiveness of the core of a belief functional and superadditivity of marginal belief functionals are jointly equivalent to MIUA. While the superadditivity was a direct application of GS's result, the exhaustiveness condition provided a novel way of comparing a decision maker's degrees of aversion to uncertainty about different sets of issues. In particular, exhaustiveness holds when the decision maker is highly averse to uncertainty about the entire set of issues collectively relative to individual issues separately. Then, we saw that MIUA has a simpler characterization and is equivalent to other notions of uncertainty aversion provided by Ghirardato and Marinacci (2002) and Chateauneuf and Tallon (2002) when uncertainty is only about the relationship between issues, but not about individual issues. We also discussed some examples of utility functions consistent with MIUA and the implications of our analysis for under-diversification.

Our study suggests several possible future studies in this vein. For example, it will be useful to understand how an analyst can identify the set of issues from a decision maker's choice behavior when only an abstract set of states without a product structure is given. Even though we have taken the set of issues as exogenous, it is completely plausible that different decision makers think of different issues given the same set of states. Studies on the identification of issues could help understand how people approach decision problems and form beliefs under uncertainty. Another future study is to parameterize the entire set of belief functionals characterized in this paper. If the set allows a handy utility functional form, it will become easier to incorporate Multi-Issue Uncertainty Aversion or under-diversifying behavior into economic models.

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A Axiomatization of the Invariant Biseparable Representation

In this section, we introduce and discuss the axiomatic foundation of the invariant biseparable representation which is provided by GMM. We list the axioms and representation result consecutively.

Axiom 1 (Weak Order). *The preference relation $\succsim$ is complete and transitive.*

Axiom 2 (Certainty Independence). *For any acts $f, g \in \mathcal{F}$, any lottery $x \in \mathcal{L}$, and any $\alpha \in (0, 1]$,*

$$f \succsim g \quad \textit{if and only if} \quad \alpha f + (1-\alpha)\bar{x} \succsim \alpha g + (1-\alpha)\bar{x}\,.$$

Axiom 3 (Archimedean Continuity). *For any acts $f, g, h \in \mathcal{F}$, if $f \succ g$ and $g \succ h$, then there exist $\alpha, \beta \in (0, 1)$ such that $\alpha f + (1-\alpha)h \succ g$ and $g \succ \beta f + (1-\beta)h$.*

Axiom 4 (Monotonicity). *For any acts $f, g \in \mathcal{F}$, if $\overline{f(s)} \succsim \overline{g(s)}$ for all $s \in S$, then $f \succsim g$.*

Axiom 5 (Nondegeneracy). *There exist acts $f, g \in \mathcal{F}$ such that $f \succ g$.*

Proposition A.1 (Ghirardato, Maccheroni, and Marinacci, 2004). *The preference relation $\succsim$ satisfies Axioms 1-5 if and only if there exists an invariant biseparable representation $(u, \mathbb{B})$ of $\succsim$. Moreover, the belief functional $\mathbb{B}$ is unique and the utility index u is unique up to a positive affine transformation.*

Axioms 1, 3, 4, and 5 are standard. Axiom 2 is a weakening of the Independence axiom. A rationale for Certainty Independence is that a mixture with a constant act is not likely to reverse a decision maker's preference as it generates no effect regarding the uncertainty over states. For example, GS argues that a mixture of two different acts may generate a hedging effect, but *not* if one of them is a constant act. We emphasize that the axiom is still appealing in our multi-issue environment, too. This is because a mixture of an act with another constant act does not change the set of relevant issues. In other words, if f is a J-act and g is a J'-act, then the mixture of them with a constant act $\bar{x}$ is still a J- and J'-act, respectively. Without any effect in terms of relevant issues, a mixture with a constant act is likely to preserve a decision maker's preference.

B Proofs of the Results in the Main Text

B.1 Preliminary results

Lemma B.1. *Assume the preference relation* $\succsim$ *satisfies Axioms 1-5 and let* $(u, \mathbb{B})$ *be an invariant biseparable representation of* $\succsim$. *Suppose that an act* $f \in \mathcal{F}$ *satisfies* $u(f) \in int(u(\mathcal{F}))$ *and that a probability* $P \in \Delta(S)$ *satisfies*

$$P \cdot u(g) \geq P \cdot u(f) \quad \text{for all } g \succsim f.$$

Then, $P \cdot u(f) = \mathbb{B}(u(f))$ *and* $P \in core(\mathbb{B})$.

Proof. Fix an act $f \in \mathcal{F}$ and a probability $P \in \Delta(S)$ that satisfies the supposition. Let $v_f = \mathbb{B}(u(f))$ and $x_f \in \mathcal{L}$ be a lottery satisfying $u(x_f) = v_f$. Then, we have

$$v_f = u(x_f) = P \cdot u(\overline{x_f}) \geq P \cdot u(f)\,, \tag{B.1}$$

where the last inequality holds since $\overline{x_f} \sim f$. Moreover, since $u(f) \in int(u(\mathcal{F}))$, there exist $g \in \mathcal{F}$ and $\alpha \in (0,1)$ such that $\alpha u(g) + (1-\alpha)\overline{v_f} = u(f)$. Applying the functional $\mathbb{B}$ on both sides, we obtain $\alpha \mathbb{B}(u(g)) + (1-\alpha) v_f = \mathbb{B}(u(f)) = v_f$, which implies $\mathbb{B}(u(g)) = v_f$, hence $g \sim f$. Thus, by the supposition,

$$P \cdot u(g) \geq P \cdot u(f)\,. \tag{B.2}$$

To show that $v_f = P \cdot u(f)$, suppose to the contrary that $v_f \neq P \cdot u(f)$. This means $v_f > P \cdot u(f)$ by (B.1). Then,

$$\begin{aligned}
P \cdot u(g) &= \frac{1}{\alpha}\Big[P \cdot u(f) - (1-\alpha) P \cdot \overline{v_f}\Big] \\
&= \frac{1}{\alpha}\Big[P \cdot u(f) - (1-\alpha) v_f\Big] \\
&< \frac{1}{\alpha}\Big[P \cdot u(f) - (1-\alpha) P \cdot u(f)\Big] \\
&= P \cdot u(f)\,,
\end{aligned}$$

which contradicts (B.2). Therefore, $\mathbb{B}(u(f)) = v_f = P \cdot u(f)$.

Next, we show that $P \in core(\mathbb{B})$. Suppose to the contrary that $P \notin core(\mathbb{B})$. Then, there exists an act $h \in \mathcal{F}$ such that $v_h := \mathbb{B}(u(h)) > P \cdot u(h)$. Since $u(f) \in int(u(\mathcal{F}))$, there exists a lottery $y \in \mathcal{L}$ and $\beta \in (0,1)$ such that

$$\begin{cases} u(y) < v_f & \text{if } h \succ f \\ u(y) = v_f & \text{if } h \sim f \\ u(y) > v_f & \text{if } h \prec f \end{cases},$$

and $h' := \beta h + (1-\beta)\overline{y} \sim f$. Let $x_h \in L$ be a lottery such that $\overline{x_h} \sim h$. Then, by Certainty Independence (Axiom 2), we obtain $\beta\overline{x_h} + (1-\beta)\overline{y} \sim h' \sim f$. Applying $\mathbb{B} \circ u$, we have

$$\beta v_h + (1-\beta)u(y) = v_f. \tag{B.3}$$

Moreover, since $h' \sim f$,

$$P \cdot u(h') \geq P \cdot u(f) = v_f\,, \tag{B.4}$$

where the second equality was shown in the earlier part of this proof. Combining (B.3) and (B.4), we obtain

$$\begin{aligned} v_f &= \beta v_h + (1-\beta)u(y) > \beta P \cdot u(h) + (1-\beta)u(y) \\ &= P \cdot u(\beta h + (1-\beta)\overline{y}) = P \cdot u(h') \geq v_f\,, \end{aligned}$$

which is a contradiction. Therefore, $P \in \mathit{core}(\mathbb{B})$. □

The following lemma shows that the converse of Lemma B.1 is also true. That is, if $P \in \mathit{core}(\mathbb{B})$ and $P \cdot u(f) = \mathbb{B}(u(f))$, then P 'supports' the upper contour set of f.

Lemma B.2. *Assume the preference relation $\succsim$ satisfies Axioms 1-5 and let $(u, \mathbb{B})$ be an invariant biseparable representation of $\succsim$. Suppose $f \in \mathcal{F}$ and $P \in \mathit{core}(\mathbb{B})$ satisfy $P \cdot u(f) = \mathbb{B}(u(f))$. Then, for any $g \in \mathcal{F}$ such that $g \succsim f$, $P \cdot u(g) \geq P \cdot u(f)$.*

Proof. Suppose $g \succsim f$. Then, $P \cdot u(g) \geq \mathbb{B}(u(g)) \geq \mathbb{B}(u(f)) = P \cdot u(f)$, where the first inequality holds since $P \in \mathit{core}(\mathbb{B})$. This concludes the proof. □

Definition B.1 (Strong Local Preference for Hedging)**.** Let $J \subset \mathcal{I}$ be a nonempty subset of issues. Then, the preference relation $\succsim$ exhibits *Strong Local Preference for Hedging (SLPH) with respect to J* if the following holds: For any pairwise indifferent acts $f_1, \cdots, f_n \in \mathcal{F}$ and any $\alpha_1, \cdots, \alpha_n \in [0,1]$ with $\sum_{k=1}^n \alpha_k = 1$, if the mixed act $g := \alpha_1 f_1 + \cdots + \alpha_n f_n$ satisfies $g(s_J, s_{-J}) \sim g(s_J, s'_{-J})$ for all $s_J \in S_J$ and $s_{-J}, s'_{-J} \in S_{\mathcal{I}\setminus J}$, then

$$\alpha_1 f_1 + \cdots + \alpha_n f_n \succsim f_1\,.$$

Lemma B.3. *Assume the preference relation $\succsim$ satisfies Axioms 1-5 and let $J \subset \mathcal{I}$ be a nonempty subset. Then, $\succsim$ exhibits Local Preference for Hedging with respect to J if and only if it exhibits Strong Local Preference for Hedging with respect to J.*

Proof. It is immediate that SLPH implies LPH. To prove the converse, assume LPH holds and let $f_1, \cdots, f_n \in \mathcal{F}$ and $\alpha_1, \cdots, \alpha_n \in [0,1]$ satisfying the supposition in SLPH be given. Write $g = \alpha_1 f_1 + \cdots + \alpha_n f_n$. It is easy to see that if all lotteries in $\{f_k(s) \in \mathcal{L} : 1 \le k \le n, s \in S\}$ are indifferent, then $g \sim f_1$ and we are done. Assume they are not all indifferent. Then, we can take two lotteries $x \succ y \in \mathcal{L}$ such that $\bar{x} \succsim \overline{f_k(s)} \succsim \bar{y}$ for all $k \in \{1, \cdots, n\}$ and $s \in S$. Then, for each k and s, there exists unique $\beta_s^k \in [0,1]$ such that $\beta_s^k \bar{x} + (1-\beta_s^k)\bar{y} \sim \overline{f_k(s)}$. For each $k \in \{1, \cdots, n\}$, define an act $h_k \in \mathcal{F}$ by $h_k(s) = \beta_s^k x + (1-\beta_s^k) y$ for all $s \in S$. Then, since $\overline{f_k(s)} \sim \overline{h_k(s)}$ for all $s \in S$, Monotonicity (Axiom 4) implies $f_k \sim h_k$ for each k, and hence $f_1 \sim \cdots \sim f_n \sim h_1 \sim \cdots \sim h_n$. Moreover, Certainty Independence (Axiom 2) implies

$$\overline{g(s)} = \overline{\textstyle\sum_{k=1}^n \alpha_k f_k(s)} \sim \overline{\textstyle\sum_{k=1}^n \alpha_k h_k(s)} \quad \forall s \in S\,. \tag{B.5}$$

Since $\overline{g(s_J, s_{-J})} \sim \overline{g(s_J, s'_{-J})}$ for all $s_J \in S_J$, $s_{-J}, s'_{-J} \in S_{\mathcal{I}\setminus J}$ by supposition, (B.5) implies

$$\overline{\textstyle\sum_{k=1}^n \alpha_k h_k(s_J, s_{-J})} \sim \overline{\textstyle\sum_{k=1}^n \alpha_k h_k(s_J, s'_{-J})} \qquad \forall s_J \in S_J, \forall s_{-J}, s'_{-J} \in S_{\mathcal{I}\setminus J}\,. \tag{B.6}$$

Since each lottery in (B.6) is a mixture of x and y, the indifference implies

$$\sum_{k=1}^n \alpha_k h_k(s_J, s_{-J}) = \sum_{k=1}^n \alpha_k h_k(s_J, s'_{-J}) \qquad \forall s_J \in S_J, \forall s_{-J}, s'_{-J} \in S_{\mathcal{I}\setminus J}\,.$$

Thus, $\sum_{k=1}^n \alpha_k h_k \in \mathcal{F}_J$. Therefore, we obtain

$$g \sim \sum_{k=1}^n \alpha_k h_k \succsim h_1 \sim f_1\,,$$

where the first relation follows from (B.5) and Monotonicity (Axiom 4), and the second relation holds by LPH. This completes the proof. □

We prove the following lemma using a similar argument as is used in the proof of Lemma 21 of Grant and Polak (2013).

Lemma B.4. *Assume the preference relation $\succsim$ satisfies Axioms 1-5 and let $(u, \mathbb{B})$ be an invariant biseparable representation of $\succsim$. If $\succsim$ additionally satisfies LPH with respect to some $J \subset \mathcal{I}$, then for any $f \in \mathcal{F}_J$ with $u(f) \in int(u(\mathcal{F}))$, there exists a probability $P \in core(\mathbb{B})$ such that $g \succsim f$ implies $P \cdot u(g) \ge P \cdot u(f)$. In this case, $P \cdot u(f) = \mathbb{B}(u(f))$.*

Proof. Fix $f \in \mathcal{F}_J$ such that $u(f) \in int(u(\mathcal{F}))$. Consider a set

$$\mathcal{U} = \big\{u(f') \in \mathbb{R}^S : f' \succsim f\big\}.$$

and the convex hull $co(\mathcal{U}) \subset \mathbb{R}^S$ of $\mathcal{U}$. The set $\mathcal{U}$ is the upper contour set of $u(f)$, or the set of all utility profiles that are preferred to $u(f)$. Clearly, $u(f) \in co(\mathcal{U})$. To see that $int(co(\mathcal{U}))$ is nonempty, note that $u(f) + \bar{\delta} \in int(u(\mathcal{F}))$, for sufficiently small $\delta > 0$, since $u(f) \in int(u(\mathcal{F}))$. By continuity of $\mathbb{B}$, any point near $u(f) + \delta$ has a higher value than $u(f)$ under $\mathbb{B}$. So, $u(f) + \bar{\delta} \in int(\mathcal{U}) \subset int(co(\mathcal{U}))$.

Now we claim that $u(f) \notin int(co(\mathcal{U}))$. Suppose $u(f)$ belongs to the interior of $co(\mathcal{U})$. Then, for sufficiently small $\epsilon > 0$, $u(f) - \bar{\epsilon} \in co(\mathcal{U})$. Moreover, we can find a J-act $g \in \mathcal{F}_J$ such that $u(g) = u(f) - \bar{\epsilon}$ and $f \succ g$. By Carathéodory's Theorem (Rockafellar, 1970, Theorem 17.1), there exist finitely many acts $h_1, \cdots, h_n \in \mathcal{F}$ and weights $\alpha_1, \cdots, \alpha_n \in (0, 1]$ with $\sum_{k=1}^n \alpha_k = 1$ such that $u(h_k) \in \mathcal{U}$, for each k, and $\alpha_1 u(h_1) + \cdots + \alpha_n u(h_n) = u(g)$. Note that n cannot be 1 since if it were, we would have $u(h_1) = u(g)$ while $u(h_1) \in \mathcal{U}$ and $u(g) \notin \mathcal{U}$. Since $h_k \succsim f \succ g$ for each k, there exist $\beta_1, \cdots, \beta_n \in (0, 1]$ such that $h'_k := \beta_k h_k + (1 - \beta_k) g \sim f$. Given this, define $\gamma_k \in (0, 1)$ by

$$\gamma_k = \frac{1}{\Gamma} \cdot \frac{\alpha_k}{\beta_k}$$

for each $k = 1, \cdots, n$, where $\Gamma = \sum_{k=1}^n \alpha_k / \beta_k$. Clearly, $\sum_{k=1}^n \gamma_k = 1$. Moreover,

$$\begin{aligned}
\sum_{k=1}^n \gamma_k u(h'_k) &= \sum_{k=1}^n \gamma_k \big(\beta_k u(h_k) + (1 - \beta_k) u(g)\big) \\
&= \frac{1}{\Gamma} \sum_{k=1}^n \Big(\alpha_k u(h_k) + \frac{\alpha_k (1 - \beta_k)}{\beta_k} u(g)\Big) \\
&= \frac{1}{\Gamma} \Big(1 + \sum_{k=1}^n \frac{\alpha_k}{\beta_k} - \sum_{k=1}^n \alpha_k\Big) u(g) \\
&= u(g)\,.
\end{aligned} \tag{B.7}$$

This equality implies that for any $s_J \in S_J$ and $s_{-J}, s'_{-J} \in S_{\mathcal{I} \setminus J}$,

$$\overline{\textstyle\sum_{k=1}^n \gamma_k h'_k(s_J, s_{-J})} \sim \overline{g(s_J, s_{-J})} = \overline{g(s_J, s'_{-J})} \sim \overline{\textstyle\sum_{k=1}^n \gamma_k h'_k(s_J, s'_{-J})}\,,$$

where the equality holds by $g \in \mathcal{F}_J$. Since SLPH holds as is shown in Lemma B.3, we obtain $\sum_{k=1}^n \gamma_k h'_k \succsim h_1$. However, this leads to $f \succ g \sim \sum_{k=1}^n \gamma_k h'_k \succsim h_1 \sim f$, where the first indifference is implied by (B.7). This is a contradiction. Thus, $u(f) \notin int(co(\mathcal{U}))$.

Hence, by the Separating Hyperplane Theorem (Aliprantis and Border, 2006, Theorem 5.67), there exists nonzero $P \in \mathbb{R}^S$ such that $\sum_{s\in S} P(s) = 1$ and $P \cdot v \geq P \cdot u(f)$ for all $v \in co(\mathcal{U})$. In particular, $f' \succsim f$ implies $u(f') \in \mathcal{U}$ and hence $P \cdot u(f') \geq P \cdot u(f)$. Thus, the functional P is the desired one we have been seeking if it belongs to $core(\mathbb{B})$. Since Monotonicity (Axiom 4) holds and $u(f) \in int(u(\mathcal{F}))$, $P(s)$ is nonnegative for each $s \in S$, which implies $P \in \Delta(S)$. Therefore, by Lemma B.1, $P \in core(\mathbb{B})$. The last statement that $P \cdot u(f) = \mathbb{B}(u(f))$ is also implied by Lemma B.1. □

Definition B.2 (Strong Multi-issue Uncertainty Aversion)**.** The preference relation $\succsim$ is said to exhibit *Strong Multi-issue Uncertainty Aversion (SMIUA)* if the following holds: For any distinct issues $i_1, \cdots, i_m \in \mathcal{I}$, any acts $f_1 \in \mathcal{F}_{i_1}, \cdots, f_m \in \mathcal{F}_{i_m}$, $g_1, \cdots, g_n \in \mathcal{F}$, and any weights $\alpha_1, \cdots, \alpha_m; \beta_1, \cdots, \beta_n \in [0, 1]$ with $\sum_{k=1}^m \alpha_k = 1$ and $\sum_{l=1}^n \beta_l = 1$, if $f_k \sim f_{k'}$ and $g_l \sim g_{l'}$ for all k, k', l, l', then

$$\overline{(\alpha_1 f_1 + \cdots + \alpha_m f_m)(s)} \sim \overline{(\beta_1 g_1 + \cdots + \beta_n g_n)(s)}, \quad \forall s \in S \quad \text{implies} \quad f_1 \succsim g_1 .$$

Lemma B.5. *Assume the preference relation $\succsim$ satisfies Axioms 1-5. Then, $\succsim$ exhibits Multi-Issue Uncertainty Aversion if and only if it exhibits Strong Multi-issue Uncertainty Aversion.*

Proof. It is immediate that SMIUA implies MIUA. We will prove the opposite direction using a similar argument with that in the proof of Lemma B.3. Let $f_1 \in \mathcal{F}_{i_1}, \cdots, f_m \in \mathcal{F}_{i_m}$, $h_1, \cdots, h_n \in \mathcal{F}$, and $\alpha_1, \cdots, \alpha_m, \beta_1, \cdots, \beta_n \in [0, 1]$ be given and suppose they satisfy the supposition of Definition B.2 for SMIUA. We need to show $f_1 \succsim h_1$. Consider the set of lotteries $M = \{f_k(s) : 1 \leq k \leq m, s \in S\} \cup \{h_l(s) : 1 \leq l \leq n, s \in S\}$. If all lotteries in M are indifferent, then by Monotonicity (Axiom 4), $f_1 \sim h_1$ and we are done. Assume they are not all indifferent. Then, there exist lotteries $x, y \in L$ such that $\bar{x} \succ \bar{y}$ and $\bar{x} \succsim \overline{x'} \succsim \bar{y}$ for every $x' \in M$. Let $\gamma_s^k, \delta_s^l \in [0, 1]$ be the unique numbers such that $\gamma_s^k \bar{x} + (1 - \gamma_s^k)\bar{y} \sim \overline{f_k(s)}$ and $\delta_s^l \bar{x} + (1 - \delta_s^l)\bar{y} \sim \overline{h_l(s)}$, for all k, l and for all $s \in S$. Then, define acts f'_k, h'_l, for each k and l, by

$$f'_k(s) = \gamma_s^k x + (1 - \gamma_s^k) y , \quad h'_l(s) = \delta_s^l x + (1 - \delta_s^l) y .$$

By Monotonicity (Axiom 4), the acts f'_k, h'_l defined above satisfy $f_1 \sim \cdots f_m \sim f'_m \sim \cdots \sim f'_1$ and $h_1 \sim \cdots \sim h_n \sim h'_n \sim \cdots \sim h'_1$. Moreover, Certainty Independence (Axiom 2) implies that, for each $s \in S$,

$$\overline{\textstyle\sum_{k=1}^m \alpha_k f'_k(s)} \sim \overline{\textstyle\sum_{k=1}^m \alpha_k f_k(s)} \sim \overline{\textstyle\sum_{k=1}^n \beta_k h_k(s)} \sim \overline{\textstyle\sum_{k=1}^n \beta_k h'_k(s)} ,$$

where the second indifference relation holds by the supposition of Definition B.2. Identified as lotteries, the leftmost and rightmost objects are mixtures of x and y. Hence, the indifference further implies that they are equal. In addition, for each $k = 1, \cdots, m$ and for all $s_{i_k} \in S_{i_k}$ and $s_{-i_k}, s'_{-i_k} \in S_{\mathcal{I}\setminus\{i_k\}}$, we have

$$\overline{f'(s_{i_k}, s_{-i_k})} \sim \overline{f(s_{i_k}, s_{-i_k})} = \overline{f(s_{i_k}, s'_{-i_k})} \sim \overline{f'(s_{i_k}, s'_{-i_k})}\,.$$

Again, since the first and last objects are mixtures of x and y, the indifference implies equality. Hence, $f'_k \in \mathcal{F}_{i_k}$ for all k. Thus, $f'_1 \in \mathcal{F}_{i_1}, \cdots, f'_m \in \mathcal{F}_{i_m}$, $h'_1, \cdots, h'_n \in \mathcal{F}$, and $\alpha_1, \cdots, \alpha_m, \beta_1, \cdots, \beta_n \in [0, 1]$ satisfy the supposition of Definition 4 for MIUA. This gives $f'_1 \succsim h'_1$. Since $f'_1 \sim f_1$ and $h'_1 \sim h_1$, we conclude that $f_1 \succsim h_1$ and $\succsim$ exhibits SMIUA. □

The following property is a stronger version of Preference for Sure Diversification of Chateauneuf and Tallon (2002), and this version is also introduced in their paper.

Definition B.3. The preference relation $\succsim$ exhibits *Preference for Sure Expected Utility Diversification (PSEUD)* if for any pairwise indifferent acts $f_1, \cdots, f_n \in \mathcal{F}$ and any $\alpha_1, \cdots, \alpha_n \in [0, 1]$ with $\sum_{k=1}^{n} \alpha_k = 1$, if the mixed act $g := \alpha_1 f_1 + \cdots + \alpha_n f_n$ satisfies $\overline{g(s)} \sim \overline{g(s')}$ for all $s, s' \in S$, then

$$\alpha_1 f_1 + \cdots + \alpha_n f_n \succsim f_1\,.$$

As Preference for Sure Diversification can be regarded as LPH with respect to the empty set, PSEUD can be regarded as SLPH with respect to the empty set. The following lemma says that PSD and PSEUD are are equivalent for invariant biseparable preferences.

Lemma B.6. *Assume the preference relation $\succsim$ satisfies Axioms 1-5. Then, $\succsim$ exhibits Preference for Sure Diversification if and only if it exhibits Preference for Sure Expected Utility Diversification.*

The proof of Lemma B.6 is essentially the same with that of Lemma B.3 except that we take $J = \varnothing$, and hence omitted. We also obtain the following lemma which is similar to Lemma B.4.

Lemma B.7. *Assume the preference relation $\succsim$ satisfies Axioms 1-5, and let $(u, \mathbb{B})$ be an invariant biseparable representation of $\succsim$. If $\succsim$ additionally exhibits Preference for Sure Diversification, then for any $f \in \mathcal{F}_\varnothing$ with $u(f) \in int(u(\mathcal{F}))$, there exists a probability $P \in core(\mathbb{B})$ such that $g \succsim f$ implies $P \cdot u(g) \geq P \cdot u(f)$.*

The proof of Lemma B.7 is also omitted since it is a slight modification of the proof of Lemma B.4 with $J = \varnothing$ taken. The last preliminary result follows below. The equivalence between (1), (2), and (3) in the statement is a result of GS.

Lemma B.8. *Suppose that $(u, \mathbb{B})$ is an invariant biseparable representation of the preference relation $\succsim$, and let J be a nonempty subset of $\mathcal{I}$. If $\succsim$ exhibits Local Preference for Hedging with respect to J, then the following equivalent conditions hold:*

(1) For any $f, g \in \mathcal{F}_J$ and $\alpha \in [0,1]$, $f \sim g$ implies $\alpha f + (1-\alpha) g \succsim f$;

(2) The marginal belief functional $\mathbb{B}_J$ is superadditive: For any $v, w \in \mathbb{R}^{S_J}$,

$$\mathbb{B}_J(v + w) \geq \mathbb{B}_J(v) + \mathbb{B}_J(w)\,;$$

(3) For any $f, g \in \mathcal{F}_J$, $f \succsim g$ if and only if

$$\min_{p \in mc_J(\mathbb{B})} p \cdot u(\varphi_J(f)) \geq \min_{p \in mc_J(\mathbb{B})} p \cdot u(\varphi_J(g)).$$

B.2 Proofs of the results in Section 4

We prove Lemma 1 and our characterization theorem. Reversing the order of presentation, we show Theorem 2 first and then Theorem 1.

B.2.1 Proof of Lemma 1

Proof. Let J be a nonempty subset of $\mathcal{I}$. Suppose $P \in core(\mathbb{B})$. We need to show that

$$marg_J(P) \cdot v \geq \mathbb{B}_J(v)\,, \quad \forall v \in \mathbb{R}^{S_J}. \tag{B.8}$$

Since $\mathbb{B}_J(v) = \mathbb{B}(\varphi_J^{-1}(v))$ by definition of $\mathbb{B}_J$ and φ_J is a bijection from $\mathcal{R}_J$ to $\mathbb{R}^{S_J}$, (B.8) is equivalent to

$$P \cdot v \geq \mathbb{B}(v)\,, \quad \forall v \in \mathcal{R}_J\,.^{36} \tag{B.9}$$

Since $\mathcal{R}_J \subset \mathbb{R}^S$ and $P \in core(\mathbb{B})$, (B.9) holds, which completes the proof. □

B.2.2 Proof of Theorem 2

Proof. We first prove the necessity of LPH. Let a nonempty subset $J \subset \mathcal{I}$ be given. Assume $\mathbb{B}_J$ is superadditive and $core(\mathbb{B})$ is J-exhaustive. Fix $f_1, \cdots, f_n \in \mathcal{F}$ and $\alpha_1, \cdots, \alpha_n \in [0,1]$, and suppose $f_1 \sim \cdots \sim f_n$, $\sum_{k=1}^n \alpha_k = 1$, and $\sum_{k=1}^n \alpha_k f_k \in \mathcal{F}_J$. Superadditivity of $\mathbb{B}_J$ and Lemma B.8 imply that $mc_J(\mathbb{B})$ is nonempty. Fix $p_J \in mc_J(\mathbb{B})$. By J-exhaustiveness, there

[36] Recall that $\mathcal{R}_J$ is the set of all utility acts measurable with respect to $\mathcal{A}_J$ as defined in (4).

exists $P \in core(\mathbb{B})$ such that $marg_J(P) = p_J$. Then, we have

$$\mathbb{B}(u(f_1)) = \sum_{k=1}^{n} \alpha_k \mathbb{B}(u(f_k)) \leq \sum_{k=1}^{n} \alpha_k P \cdot u(f_k)\,.$$

By the affinity of u,

$$\sum_{k=1}^{n} \alpha_k P \cdot u(f_k) = P \cdot u\Big(\sum_{k=1}^{n} \alpha_k f_k\Big)\,.$$

Since $\sum_{k=1}^{n} \alpha_k f_k \in \mathcal{F}_J$,

$$P \cdot u\Big(\sum_{k=1}^{n} \alpha_k f_k\Big) = p_J \cdot u\Big(\varphi_J\big(\sum_{k=1}^{n} \alpha_k f_k\big)\Big)\,.$$

Since p_J is arbitrary, it follows from Lemma B.8 that

$$\mathbb{B}(u(f_1)) \leq \min_{p_J \in mc_J(\mathbb{B})} p_J \cdot u\Big(\varphi_J\big(\sum_{k=1}^{n} \alpha_k f_k\big)\Big) = \mathbb{B}\Big(u\big(\sum_{k=1}^{n} \alpha_k f_k\big)\Big)\,.$$

Therefore, $\sum_{k=1}^{n} \alpha_k f_k \succsim f_1$.

We turn to the proof of sufficiency of LPH. Superadditivity of $\mathbb{B}_J$ is immediate from Lemma B.8. We need to show J-exhaustiveness of $core(\mathbb{B})$. Suppose to the contrary that there exists a probability $q_J \in mc_J(\mathbb{B})$ such that $q_J \notin marg_J(core(\mathbb{B}))$. Assume without loss of generality that $0 \in int(u(\mathcal{L}))$. Then, since $marg_J(core(\mathbb{B}))$ is closed and convex, by the Separating Hyperplane Theorem (Aliprantis and Border, 2006, Corollary 5.80), there exists $v \in int\big(u(\varphi_J(\mathcal{F}_J))\big) \subset \mathbb{R}^{S_J}$ such that $p_J \cdot v > q_J \cdot v$ for all $p_J \in marg_J(core(\mathbb{B}))$. Let $f \in \mathcal{F}_J$ be a J-act that satisfies $u(\varphi_J(f)) = v$. Then,

$$\mathbb{B}(u(f)) = \min_{\tilde{p}_J \in mc_J(\mathbb{B})} \tilde{p}_J \cdot v \leq q_J \cdot v < p_J \cdot v\,, \quad \forall p_J \in marg_J(core(\mathbb{B}))\,. \tag{B.10}$$

Moreover, by Lemma B.4, there exists a probability $Q \in core(\mathbb{B})$ such that $Q \cdot u(f) = \mathbb{B}(u(f))$. Since $marg_J(Q) \in marg_J(core(\mathbb{B}))$, we obtain

$$\mathbb{B}(u(f)) = Q \cdot u(f) = marg_J(Q) \cdot u(\varphi_J(f)) = marg_J(Q) \cdot v > \mathbb{B}(u(f))\,,$$

where the last inequality follows from (B.10). This is a contradiction. Therefore, we have proved that $core(\mathbb{B})$ is J-exhaustive. □

B.2.3 Proof of Theorem 1

Proof. **(Necessity)** First, we show the necessity of MIUA. Fix distinct issues $i_1, \cdots, i_m \in \mathcal{I}$. Let $f_1 \in \mathcal{F}_{i_1}, \cdots, f_m \in \mathcal{F}_{i_m}$, $g_1, \cdots, g_n \in \mathcal{F}$, and $\alpha_1, \cdots, \alpha_m, \beta_1, \cdots, \beta_n \in [0,1]$ be given. Suppose $f_1 \sim \cdots \sim f_m$, $g_1 \sim \cdots \sim g_n$, and $\sum_{k=1}^m \alpha_k f_k = \sum_{l=1}^n \beta_l g_l$. We need to show that $f_1 \succsim g_1$. Since $\mathbb{B}_{i_k}$ is superadditive for each $k = 1, \cdots, m$, Lemma B.8 implies that there exist $p_1 \in mc_{i_1}(\mathbb{B}), \cdots, p_m \in mc_{i_m}(\mathbb{B})$ such that $\mathbb{B}(u(f_k)) = p_k \cdot u(\varphi_{i_k}(f_k))$ for all k. Since $core(\mathbb{B})$ is exhaustive, there exists $P \in core(\mathbb{B})$ such that $marg_{i_k}(P) = p_k$ for each k. Then, for each $k = 1, \cdots, n$,

$$P \cdot u(f_k) = marg_{i_k}(P) \cdot u(\varphi_{i_k}(f_k)) = \mathbb{B}(u(f_k)) \,. \tag{B.11}$$

In particular, $P \cdot u(f_1) = \mathbb{B}(u(f_1))$. Hence, by Lemma B.2, the half space $\mathcal{H} = \{v \in \mathbb{R}^S : P \cdot v \geq P \cdot u(f_1)\}$ contains $\mathcal{U} = \{u(f') \in \mathbb{R}^S : f' \succsim f\}$. Moreover, (B.11) also implies that $P \cdot u(f_1) = \cdots = P \cdot u(f_k)$. Thus, $P \cdot u(\sum_{k=1}^m \alpha_k f_k) = \sum_{k=1}^m \alpha_k P \cdot u(f_k) = P \cdot u(f_1)$, which implies that $u(\sum_{k=1}^m \alpha_k f_k)$ is a boundary point of the half space $\mathcal{H}$.

Given this, suppose to the contrary that $g_1 \succ f_1$. Then, by continuity of $\mathbb{B}$, $u(g_1), \cdots, u(g_n)$ belong to the interior of $\mathcal{U}$, and hence to the interior of $\mathcal{H}$. Thus, we have $u\big(\sum_{l=1}^n \beta_l g_l\big) = \sum_{l=1}^n \beta_l u(g_l) \in int(\mathcal{H})$. This contradicts $\sum_{k=1}^m \alpha_k f_k = \sum_{l=1}^n \beta_l g_l$ since $u(\sum_{k=1}^m \alpha_k f_k)$ lies on the boundary of $\mathcal{H}$. Therefore, $f_1 \succsim g_1$.

(Sufficiency) Suppose the preference relation $\succsim$ exhibits MIUA and write $\mathcal{I} = \{i_1, \cdots, i_m\}$, where the issues $i_1, \cdots, i_m$ are distinct. Since it implies LPH with respect to i for each $i \in \mathcal{I}$, the functional $\mathbb{B}_i$ is superadditive, for each i, by Lemma B.8. To show that $core(\mathbb{B})$ is exhaustive, let

$$M = \{(marg_{i_1}(P), \cdots, marg_{i_m}(P)) \in mc_{i_1}(\mathbb{B}) \times \cdots \times mc_{i_m}(\mathbb{B}) : P \in core(\mathbb{B})\} \,.$$

Since $core(\mathbb{B})$ is closed and convex, so is M. Suppose to the contrary that $core(\mathbb{B})$ is not exhaustive. Then, there exists $(q_1, \cdots, q_m) \in \big(\times_{k=1}^m mc_{i_k}(\mathbb{B})\big) \setminus M$. Using the Separating Hyperplane Theorem (Aliprantis and Border, 2006, Corollary 5.80), we can find $v = (v_1, \cdots, v_m) \in \mathbb{R}^{\cup_{k=1}^m S_{i_k}}$, with $v_k \in \mathbb{R}^{S_{i_k}}$ for each k, such that $\sum_{k=1}^m p_k \cdot v_k > \sum_{k=1}^m q_k \cdot v_k$, for all $(p_1, \cdots, p_m) \in M$. This condition can be rewritten as

$$\sum_{k=1}^m marg_{i_k}(P) \cdot v_k > \sum_{k=1}^m q_k \cdot v_k \,, \qquad \forall P \in core(\mathbb{B}). \tag{B.12}$$

Assume without loss of generality that $0 \in int(u(\mathcal{L}))$. Then, we can choose sufficiently small

$\epsilon > 0$ so that

$$\begin{aligned} &\inf u(\mathcal{L}) < \min_{1\le k\le m} \min_{s_{i_k}\in S_{i_k}} \epsilon v_k(s_{i_k})\,; \quad \text{and} \\ &\max_{1\le k\le m} \max_{s_{i_k}\in S_{i_k}} \epsilon v_k(s_{i_k}) + \max_{1\le k,k'\le m} \left|\mathbb{B}_{i_k}(\epsilon v_k) - \mathbb{B}_{i_{k'}}(\epsilon v_{k'})\right| < \sup u(\mathcal{L})\,. \end{aligned} \tag{B.13}$$

Then, for each $k = 1, \cdots, m$, let

$$c_k = \left[\max_{1\le k'\le m} \mathbb{B}_{i_{k'}}(\epsilon v_{k'})\right] - \mathbb{B}_{i_k}(\epsilon v_k)\,,$$

and

$$v'_k = \epsilon v_k + \varphi_{i_k}(\overline{c_k})\,.$$

By construction of ϵ in (B.13), $\varphi_{i_k}^{-1}(v'_k) \in int(u(\mathcal{F}_{i_k}))$ for all k. Moreover, we have

$$\mathbb{B}_{i_k}(v'_k) = \mathbb{B}_{i_k}(\epsilon v_k + \varphi_{i_k}(\overline{c_k})) = \mathbb{B}_{i_k}(\epsilon v_k) + c_k = \max_{1\le k'\le m} \mathbb{B}_{i_{k'}}(\epsilon v_{k'})\,. \tag{B.14}$$

Now for each k, let $f_k \in \mathcal{F}_{i_k}$ be an i_k-act such that $u(\varphi_{i_k}(f_k)) = v'_k$. Then, since $\mathbb{B}(u(f_k)) = \mathbb{B}_{i_k}(u(\varphi_{i_k}(f_k))) = \mathbb{B}_{i_k}(v'_k)$, we have $f_1 \sim \cdots \sim f_m$ from (B.14). Moreover, we obtain that for any $P \in core(\mathbb{B})$,

$$\begin{aligned} &\textstyle\sum_{k=1}^m marg_{i_k}(P)\cdot u(\varphi_{i_k}(f_k)) = \sum_{k=1}^m marg_{i_k}(P)\cdot v'_k \\ &= \textstyle\sum_{k=1}^m marg_{i_k}(P)\cdot(\epsilon v_k + \varphi_{i_k}(\overline{c_k})) = \sum_{k=1}^m \left(marg_{i_k}(P)\cdot \epsilon v_k + c_k\right) \\ &> \textstyle\sum_{k=1}^m \left(q_k\cdot \epsilon v_k + c_k\right) = \sum_{k=1}^m q_k\cdot(\epsilon v_k + \varphi_{i_k}(\overline{c_k})) \\ &= \textstyle\sum_{k=1}^m q_k\cdot v'_k = \sum_{k=1}^m q_k\cdot u(\varphi_{i_k}(f_k))\,, \end{aligned} \tag{B.15}$$

where the inequality holds by (B.12).

Let $\mathcal{U} = \{u(f') \in \mathbb{R}^S : f' \succsim f_1\}$. We claim that

$$co\{u(f_k) : 1 \le k \le m\} \cap int(co(\mathcal{U})) = \varnothing \tag{B.16}$$

Suppose not. Then, there exist some $\tilde{\alpha}_1, \cdots, \tilde{\alpha}_m \in [0,1]$ with $\sum_{k=1}^m \tilde{\alpha}_k = 1$ such that $u(\sum_{k=1}^m \tilde{\alpha}_k f_k) \in int(co(\mathcal{U}))$. So, there is sufficiently small $\eta > 0$ such that $u(\sum_{k=1}^m \tilde{\alpha}_k f_k) - \bar{\eta} \in co(\mathcal{U})$ and $u(f_k) - \bar{\eta} \in u(\mathcal{F}_{i_k})$ for all k. For each k, let $\tilde{f}_k \in \mathcal{F}_{i_k}$ be an i_k-act such that $u(\tilde{f}_k) = u(f_k) - \bar{\eta}$. Then, we have $f_k \succ \tilde{f}_k$ and $u(\sum_{k=1}^m \tilde{\alpha}_k \tilde{f}_k) = u(\sum_{k=1}^m \tilde{\alpha}_k f_k) - \bar{\eta} \in co(\mathcal{U})$. The same argument in the proof of Lemma B.4 that uses Carathéodory's Theorem implies that there exist finitely many pairwise indifferent acts $h_1, \cdots, h_n \in \mathcal{F}$ such that $h_1 \sim f_1$ and $\sum_{l=1}^n \gamma_l u(h_l) = u(\sum_{k=1}^m \tilde{\alpha}_k \tilde{f}_k)$ for some weights $\gamma_1, \cdots, \gamma_n$. Thus, we obtain $\tilde{f}_1 \succsim h_1$

from SMIUA, which is implied by MIUA (Lemma B.5). However, this is contradictory to $h_1 \sim f_1 \succ \tilde{f}_1$. Therefore, (B.16) is true.

Since $u(f_1) \in int(u(\mathcal{F}_{i_1}))$ by construction and $int(u(\mathcal{F}_{i_1})) \subset int(u(\mathcal{F}))$, there exists sufficiently small $\zeta > 0$ such that $u(f_1) + \bar{\zeta} \in int(u(\mathcal{F}))$. Since $\mathbb{B}(u(f_1) + \bar{\zeta}) > \mathbb{B}(u(f_1))$ and $\mathbb{B}$ is continuous, $u(f_1) + \bar{\zeta} \in int(\mathcal{U})$. Hence, $int(co(\mathcal{U}))$ is nonempty. Thus, the Separating Hyperplane Theorem (Aliprantis and Border, 2006, Theorem 5.67) implies that there exists nonzero $Q \in \mathbb{R}^S$ with $\sum_{s \in S} Q(s) = 1$ and $c \in \mathbb{R}$ such that $Q \cdot t \geq c$ for all $t \in co(\mathcal{U})$ and $Q \cdot t \leq c$ for all $t \in co\{u(f_k) : 1 \leq k \leq m\}$. Since $Q \cdot u(f_k) \leq c$ for each k, $co(\mathcal{U})$ is contained in the halfspace $\{t \in \mathbb{R}^S : Q \cdot t \geq Q \cdot u(f_k)\}$ for each k. Moreover, by Monotonicity (Axiom 4), $Q \in \Delta(S)$. Hence, Lemma B.1 implies that $Q \cdot u(f_k) = \mathbb{B}(u(f_k))$ for all k and $Q \in core(\mathbb{B})$. Combining these with (B.15), we obtain

$$\begin{aligned}
\textstyle\sum_{k=1}^m \mathbb{B}(u(f_k)) &= \textstyle\sum_{k=1}^m Q \cdot u(f_k) \\
&= \textstyle\sum_{k=1}^m marg_{i_k}(Q) \cdot u(\varphi_{i_k}(f_k)) \\
&> \textstyle\sum_{k=1}^m q_k \cdot u(\varphi_{i_k}(f_k)) \\
&\geq \textstyle\sum_{k=1}^m \mathbb{B}(u(f_k)),
\end{aligned}$$

where the last inequality holds since $q_k \in mc_{i_k}(\mathbb{B})$ and $\mathbb{B}_{i_k}$ is superadditive, for all k. This is a contradiction. Therefore, we conclude that $core(\mathbb{B})$ is exhaustive, and the proof of sufficiency is completed. □

B.3 Proofs of the results in Section 5

B.3.1 Proof of Lemma 2

Proof. To show that exhaustiveness implies nonemptiness, suppose $core(\mathbb{B})$ is exhaustive. Since each $\mathbb{B}_i$ is additive for each $i \in \mathcal{I}$, $mc_i(\mathbb{B})$ is a singleton, in particular nonempty. Hence, by definition of exhaustiveness, $core(\mathbb{B})$ is nonempty. To prove the converse, suppose $core(\mathbb{B})$ is nonempty. Note that $\times_{i \in \mathcal{I}} mc_i(\mathbb{B})$ is a singleton since every $\mathbb{B}_i$ is additive. Moreover, by Lemma 1, $marg_i(P)$ must be equal to the probability in $mc_i(\mathbb{B})$ for all $i \in \mathcal{I}$ and for all $P \in core(\mathbb{B})$. Therefore, $core(\mathbb{B})$ is exhaustive. □

B.3.2 Proof of Theorem 3

Proof. Let $(u, \mathbb{B})$ be an invariant biseparable representation of $\succsim$. We first show that Ambiguity Aversion and PSD, respectively, are equivalent to nonemptiness of $core(\mathbb{B})$. The equivalence between Ambiguity Aversion and nonemptiness of $core(\mathbb{B})$ is shown in Theorem 12 of Ghirardato and Marinacci (2002). Moreover, Lemma B.7 shows that PSD implies nonemptiness of $core(\mathbb{B})$. Conversely, suppose $core(\mathbb{B})$ is nonempty and let $P \in core(\mathbb{B})$. We will show that $\succsim$ exhibits PSD. Let $f_1, \cdots, f_n \in \mathcal{F}$ and $\alpha_1, \cdots, \alpha_n \in [0,1]$ with $\sum_{k=1}^{n} \alpha_k = 1$ be given, and assume $f_1 \sim \cdots \sim f_n$ and $g := \alpha_1 f_1 + \cdots + \alpha_n f_n \in \mathcal{F}_\varnothing$. Suppose to the contrary that

$$f_1 \succ g = \alpha_1 f_1 + \cdots + \alpha_n f_n .$$

Since $g \in \mathcal{F}_\varnothing$, Lemma B.2 implies that

$$P \cdot u(h) \geq P \cdot u(g) \text{ for all } h \succsim g. \tag{B.17}$$

In particular, $P \cdot u(f_1) \geq P \cdot u(g)$. Moreover, $P \cdot u(f_1) > P \cdot u(g)$ since if the equality holds, then $\mathbb{B}(u(g)) = P \cdot u(g) = P \cdot u(f_1) \geq \mathbb{B}(u(f_1))$ which contradicts $f_1 \succ g$. Since $f_1 \sim \cdots \sim f_n$, it can be shown in the same way that $P \cdot u(f_k) > P \cdot u(g)$. Then, we obtain

$$P \cdot u(g) = P \cdot u(\alpha_1 f_1 + \cdots + \alpha_n f_n) = \sum_{k=1}^{n} \alpha_k P \cdot u(f_k) > P \cdot u(g) ,$$

which is a contradiction. Thus, it must be true that $g \succsim f$. Therefore, $\succsim$ exhibits PSD.

Next, we show that MIUA implies Ambiguity Aversion and PSD. Assume MIUA. Then, for each $i \in \mathcal{I}$, $mc_i(\mathbb{B})$ is nonempty since $\mathbb{B}_i$ is superadditive by Theorem 1. Then, $core(\mathbb{B})$ is also nonempty since it is exhaustive by Theorem 1. Therefore, by the equivalence we proved in the previous paragraph, $\succsim$ exhibits Ambiguity Aversion and PSD.

Lastly, we show the equivalence between (1), (2), (3), and (4) when $\succsim$ satisfies Independence on $\mathcal{F}_i$ for each $i \in \mathcal{I}$. We already saw that (2), (3), and (4) are equivalent (even without Independence). Moreover, since each $\mathbb{B}_i$ is additive by Independence imposed on $\mathcal{F}_i$, Lemma 2 implies the equivalence between (1) and (4). So, the proof of the theorem is completed. □

B.4 Proof of the results in Section 7

B.4.1 Proof of Proposition 2

Proof. Fix distinct issues $i_1, \cdots, i_n \in \mathcal{I}$, acts $f_1 \in \mathcal{F}_{i_1}, \cdots, f_n \in \mathcal{F}_{i_n}$,, lotteries $x_1, \cdots, x_n \in \mathcal{L}$, and weights $\alpha_1, \cdots, \alpha_n \in [0,1]$. Suppose $f_k \sim \bar{x}_k$ for each k. Then, by Certainty Independence (Axiom 2), we have

$$\alpha_k f_k + \textstyle\sum_{l \neq k} \alpha_l \bar{x}_l \sim \sum_{l=1}^{n} \alpha_l \bar{x}_l \quad \forall k = 1, \cdots, n.$$

Write $g_k = \alpha_k f_k + \sum_{l \neq k} \alpha_l \bar{x}_l$. Since $g_k \in \mathcal{F}_{i_k}$ for each k and g_k's are pairwise indifferent by transitivity, MIUA $\mathcal{I}$ implies

$$g_1 \succsim \frac{1}{n} g_1 + \cdots \frac{1}{n} g_n \,.$$

By rearranging the right-hand side, we obtain

$$\textstyle\sum_{l=1}^n \alpha_l \bar{x}_l \sim g_1 \succsim \frac{1}{n}\Big(\sum_{l=1}^n \alpha_l f_l \Big) + \frac{n-1}{n}\Big(\sum_{l=1}^n \alpha_l \bar{x}_l \Big) \,.$$

Applying Certainty Independence (Axiom 2) again, we obtain

$$\textstyle\sum_{l=1}^n \alpha_l \bar{x}_l \succsim \sum_{l=1}^n \alpha_l f_l \,.$$

which is the desired result. □

B.4.2 Proof of Proposition 3

Proof. Assume $\mathcal{I} = \{1, 2\}$ and let ν_i be a convex capacity on S_i for each $i \in \mathcal{I}$. Fix $i \neq j \in \mathcal{I}$. Let i-acts $f_1, \cdots, f_{n+1} \in \mathcal{F}_i$ and pairwise disjoint j-events $E_1, \cdots, E_n \in \mathcal{A}_j$ be given. Suppose $\succsim$ is a CEU preference with a utility index u and the capacity $\pi^H(\nu)$. Write $g = f_1 E_1 \cdots f_n E_n f_{n+1}$ and suppose that $f_1, \cdots, f_{n+1}$ are pairwise indifferent. We will show that $f_1 \succsim g$.

By the indifference,

$$\int_{S_i} u\big(\varphi_i(f_k)\big)\, d\nu_i = \int_S u(f_k)\, d\pi^H(\nu) = \int_S u(f_l)\, d\pi^H(\nu) = \int_{S_i} u\big(\varphi_i(f_l)\big)\, d\nu_i \tag{B.18}$$

for all $k, l \in \{1, \cdots, n+1\}$. Since ν_i is convex, there exists $p_k \in core(\nu_i)$ such that

$$\int_{S_i} u\big(\varphi_i(f_k)\big)\, d\nu_i = \int_{S_i} u\big(\varphi_i(f_k)\big)\, dp_k \tag{B.19}$$

for each $k = 1, \cdots, n+1$. Now choose a probability $q_j \in core(\nu_j)$ and define a probability Q on S by

$$Q(s_i, s_j) = p_{k(s_j)}(s_i) q_j(s_j) \,, \quad \forall s_i \in S_i \,, \forall s_j \in S_j,$$

where $k(s_j) \in \{1, \cdots, n+1\}$ is the unique index that satisfies $(s_i, s_j) \in E_{k(s_j)}$ and $E_{n+1} = S \backslash \big(E_1 \cup \cdots \cup E_n\big)$. Write $E_k = S_i \times \tilde{E}_k$ with $\tilde{E}_k \subset S_j$ for each $k = 1, \cdots, n+1$. Then, for any rectangular event $G_i \times G_j \subset S$, we have

$$Q(G_i \times G_j) = \sum_{k=1}^{n+1} Q\big(E_k \cap (G_i \times G_j)\big)$$

$$
\begin{aligned}
&= \sum_{k=1}^{n+1} Q\big(G_i \times (\tilde{E}_k \cap G_j)\big) \\
&= \sum_{k=1}^{n+1} p_k(G_i) q_j\big(\tilde{E}_k \cap G_j\big) \\
&\geq \sum_{k=1}^{n+1} \nu_i(G_i) q_j\big(\tilde{E}_k \cap G_j\big) && (\because p_k \in core(\nu_i)) \\
&= \nu_i(G_i) q_j(G_j) && (\because \tilde{E}_1 \cup \cdots \cup \tilde{E}_{n+1} = S_j) \\
&\geq \nu_i(G_i) \nu_j(G_j)\,. && (\because q_j \in core(\nu_j))
\end{aligned}
$$

This implies $Q \in C^H(\nu) \subset core(\pi^H(\nu))$. Moreover,

$$
\begin{aligned}
\int_S u(g)\,dQ &= \sum_{k=1}^{n+1} \int_{E_k} u(f_k)\,dQ = \sum_{k=1}^{n+1} Q(E_k) \int_{S_i} u\big(\varphi_i(f_k)\big)\,dp_i \\
&= \int_{S_i} u\big(\varphi_i(f_1)\big)\,dp_i = \int_S u(f_1)\,d\pi^H(\nu)\,,
\end{aligned}
$$

where the third equality follows from (B.18) and (B.19). Therefore, we obtain $f_1 \succsim g$ from

$$
\int_S u(g)\,d\pi^H(\nu) \leq \min_{P \in core(\pi^H(\nu))} \int_S u(g)\,dP \leq \int_S u(g)\,dQ = \int_S u(f_1)\,d\pi^H(\nu)\,.
$$

$\square$

C Choquet Expected Utility

We introduce the Choquet Expected Utility, its axiomatic foundation, and some facts regarding its core in this section. We will only cover the case with a finite set of states. See the original papers of Schmeidler (1986, 1989) or other related papers (e.g., Ghirardato, 1997, 2001) for more details.

C.1 Choquet integral

Let Ω be a nonempty finite set. We start by defining a capacity.

Definition C.1. A set function $\nu : 2^\Omega \to [0, 1]$ is a *capacity* on Ω if

(1) $\nu(\varnothing) = 0$ and $\nu(\Omega) = 1$;

(2) $\nu(E) \leq \nu(F)$ for all $E \subset F \subset \Omega$.

A capacity ν is called a probability if it additionally satisfies additivity: $\nu(E \cup F) = \nu(E) + \nu(F)$ for all disjoint subsets $E, F \subset \Omega$. It is called *convex* if $\nu(E \cup F) + \nu(E \cap F) \geq \nu(E) + \nu(F)$ for all $E, F \subset \Omega$. Now we define the Choquet integral of a real-valued function with respect to a capacity. Let $\psi : \Omega \to \mathbb{R}$ be a real-valued function on Ω, and number the elements of Ω so that

$$\psi(\omega_1) \leq \cdots \leq \psi(\omega_{|\Omega|}) .$$

Then, given a capacity ν on Ω, the *Choquet integral of ψ with respect to ν* is

$$\int_\Omega \psi \, d\nu = \psi(\omega_1) + \sum_{k=2}^{|\Omega|} \big(\psi(\omega_k) - \psi(\omega_{k-1})\big) \, \nu\big(\{\omega_k, \cdots, \omega_{|\Omega|}\}\big) . \tag{C.1}$$

Equivalently,

$$\int_\Omega \psi \, d\nu = \int_0^\infty \nu\big(\{\omega \in \Omega : \psi(\omega) \geq t\}\big) \, dt + \int_{-\infty}^0 \big[\nu\big(\{\omega \in \Omega : \psi(\omega) \geq t\}\big) - 1\big] \, dt ,$$

where the integrals on the right-hand side are taken in Riemann sense.

C.2 CEU representation

We now consider our model in the main text, and present the definition of the CEU representation.

Definition C.2. A pair (u, ν) consisting of a nonconstant affine utility index $u : \mathcal{L} \to \mathbb{R}$ and a capacity ν on S is a *Choquet Expected Utility representation* of the preference relation $\succsim$ if the utility function

$$f \mapsto \int_S u(f)\, d\nu$$

represents $\succsim$.

Schmeidler (1989) provides the axiomatic foundation of the CEU representation. We say that two acts f and g are *comonotonic* if for all $s, s' \in S$, $\overline{f(s)} \succ \overline{f(s')}$ implies $\overline{g(s)} \succsim \overline{g(s')}$. The preference relation satisfies *Comonotonic Independence* if for any pairwise comonotonic acts $f, g, h \in \mathcal{F}$ and any $\alpha \in (0, 1]$, $f \succsim g$ if and only if $\alpha f + (1-\alpha)h \succsim \alpha g + (1-\alpha)h$. Clearly, *Comonotonic Independence* is a weakening of Independence. However, it is stronger than Certainty Independence (Axiom 2). It is shown by Schmeidler (1989) that the preference relation $\succsim$ satisfies Axioms 1, 3, 4, 5, and Comonotonic Independence if and only if it has a CEU representation.

C.3 Core of a capacity

The core of a capacity is defined as follows.

Definition C.3. The core of a capacity ν on S is

$$core(\nu) = \{P \in \Delta(S) : P(E) \geq \nu(E),\ \forall E \subset S\}.$$

Since the integral $\int_S \cdot\, d\nu$ is monotone and constant linear, the pair $(u, \int_S \cdot\, d\nu)$ is an invariant biseparable representation. We can show that the core of the capacity ν coincides with the core of the belief functional $\int_S \cdot\, d\nu$.

Fact C.1. *Let ν be a capacity on S, and let $\mathbb{B} : \mathbb{R}^S \to \mathbb{R}$ be the belief functional defined by $\mathbb{B}(v) = \int_S v\, d\nu$ for all $v \in \mathbb{R}^S$. Then, $core(\nu) = core(\mathbb{B})$.*

Proof. Suppose $P \in core(\nu)$. Let $v \in \mathbb{R}^S$ be given. Then, using a numbering of states in S such that $v(s_1) \leq \cdots \leq v(s_{|S|})$ as in (C.1), we have

$$\begin{aligned}\mathbb{B}(v) &= v(s_1) + \sum_{k=2}^{|S|} \big(v(s_k) - v(s_{k-1})\big)\, \nu\big(\{s_k, \cdots, s_{|S|}\}\big) \\ &\leq v(s_1) + \sum_{k=2}^{|S|} \big(v(s_k) - v(s_{k-1})\big)\, P\big(\{s_k, \cdots, s_{|S|}\}\big) = \sum_{k=1}^{|S|} v(s_k)P(s_k) = P \cdot v\,,\end{aligned} \tag{C.2}$$

where the inequality holds since $P \in core(\nu)$. This immediately implies that $P \in core(\mathbb{B})$. Thus, $core(\nu) \subset core(\mathbb{B})$.

Conversely, suppose $P \in core(\mathbb{B})$. Let $E \subset S$ be given and let $e_E \in \mathbb{R}^S$ be the vector such that $e_E(s) = 1$ if $s \in E$ and $e_E(s) = 0$ otherwise. Then,

$$\nu(E) = \int_S e_E \, d\nu = \mathbb{B}(e_E) \leq P \cdot e_E = P(E)\,.$$

Since E is arbitrarily given, $P \in core(\nu)$. Therefore, $core(\mathbb{B}) \subset core(\nu)$. □

Fact C.1 immediately implies the following.

Fact C.2. *If ν is a capacity on S and P belongs to $core(\nu)$, then $P \cdot v \geq \int_S v \, d\nu$ for all $v \in \mathbb{R}^S$.*

When a nonempty closed convex subset $C \subset \Delta(S)$ is given, we can think of two ways of taking a minimum utility. One is to consider the MEU utility: $\min_{P \in C} \int_S u(f) dP$. The other is to consider the lower envelope of C and the CEU utility with respect to it as we did in Section 6. As we claimed in the section, the latter is dominated by the former for all acts.

Fact C.3. *Let C be a closed convex subset of $\Delta(S)$, and λ be a capacity on S defined by*

$$\lambda(E) = \min_{P \in C} P(E)\,, \quad \forall E \subset S\,.$$

Then, for any utility act $v \in \mathbb{R}^S$,

$$\int_S v \, d\lambda \leq \min_{P \in C} P \cdot v\,.$$

Proof. By definition of λ, any probability in C belongs to $core(\lambda)$. Thus, by Fact C.2, for any $v \in \mathbb{R}^S$ and $P \in C$,

$$\int_S v \, d\lambda \leq P \cdot v\,.$$

Since this holds for any $P \in C$, we obtain the desired result. □

Lastly, a CEU belief functional is superadditive if and only if its associated capacity is convex. Moreover, if $\succsim$ is a CEU preference represented by (u, ν), then the following are equivalent: (1) $\succsim$ satisfies Uncertainty Aversion; (2) The capacity ν is convex; (3) The belief functional $\int_S \cdot \, d\nu$ is superadditive; (4) $\int_S u(f) \, d\nu = \min_{P \in core(\nu)} \int_S u(f) \, dP$. Schmeidler (1989) provides additional equivalent conditions (Proposition, p.582).