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Coherent Dempster–Shafer equilibrium and ambiguous signals☆

Adam Dominiak, Min Suk Lee *

Department of Economics, Virginia Tech, Blacksburg, VA 24061, USA

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ABSTRACT

This paper reappraises the Dempster–Shafer equilibrium, a novel solution concept for signaling games introduced by Eichberger and Kelsey (2004), and suggests a new refinement approach. It is demonstrated that if the types of the Sender – but not messages – are assumed to be ex-ante unambiguous, then the Receiver's conditional Choquet preference derived by the Dempster–Shafer updating rule coincides with subjective expected utility. This property of the pessimistic updating rule narrows the pooling, but not separating, Dempster–Shafer equilibrium to be behaviorally equivalent to the perfect Bayesian equilibrium. Moreover, if one refines the separating Dempster–Shafer equilibrium à la Ryan (2002a) by imposing the belief persistence axiom, then no deviations from the perfect Bayesian equilibrium are feasible. To eliminate Ryan's type of behavior, a less stringent refinement based on the notion of coherent beliefs is elaborated.

1. Introduction

Signaling games constitute a prominent class of dynamic games under incomplete information. The informed player (the Sender) sends a message contingent on her private information. The uninformed player (the Receiver) observes the message and makes an inference about the Sender's private information. Under the Bayesian paradigm, Bayes' rule is followed when updating beliefs and the perfect Bayesian equilibrium (PBE) is the standard solution concept. However, in non-Bayesian setups, messages might be ambiguous and updating rules other than Bayes' are conceivable. Eichberger and Kelsey (2004) suggested a novel equilibrium concept that incorporates a pessimistic way of updating beliefs: the Dempster–Shafer equilibrium (DSE). A primary motivation for the solution concept was to explain strategic behavior that cannot be captured by PBE. In line with this motivation, we reappraise the DSE notion as a descriptive tool and suggest a new refinement criterion.

The DSE is a solution concept for signaling games under ambiguity. Ambiguity is modeled via Choquet expected utility theory à la Schmeidler (1989). Players' beliefs are represented by capacities (i.e., non-additive probabilities). The key feature of the DSE is an updating rule for non-additive beliefs. After observing a message, the Receiver updates his ex-ante beliefs by applying the revision rule introduced by Dempster (1968) and Shafer (1976). The Dempster–Shafer updating rule, axiomatized by Gilboa and Schmeidler (1993), formalizes the idea of pessimistic belief change. That is, the message observed is always regarded as "not-good-news". The attractiveness of the pessimistic updating rule relies on the fact that the conditional capacity is well defined on the events of measure zero, provided the events (e.g., messages) are ambiguous. This property enables updating at information sets off-the-equilibrium-path.

What is our rationale for reappraising the DSE notion? Our main motivation is an intriguing observation made by Ryan (2002b). He pointed out that the DSE notion might support a "troublesome" behavior. In his game, the "troublesome" behavior refers to a "fully" separating DSE in which the Receiver cannot infer the true types, although revealing private information is the Sender's strictly dominant strategy. Ryan attributed such behavior to the violation of the so-called belief persistence axiom. Roughly speaking, the axiom requires new information to be embedded with the smallest possible change to ex-ante beliefs (see Battigalli and Bonanno, 1997). For instance, an updated capacity that attaches a positive value to the states that are not included in an ex-ante support violates the axiom. To eliminate the "troublesome" behavior, Ryan advocated to refine the DSE by imposing the belief persistence axiom on equilibrium beliefs.

As we shall argue in the following, the refinement based on the belief persistence axiom is very restrictive since the refined

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* Corresponding author.
E-mail addresses: dominiak@vt.edu (A. Dominiak), iminlee@vt.edu (M.S. Lee).

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beliefs preclude any ambiguity at every separating DSE. Moreover, although Ryan's behavior might be seen as "troublesome", it is not caused by violations of the belief persistence axiom but the fact that the DSE does not impose any "rationality" condition on the Receiver's conditional beliefs. To rectify the problem, we refine the DSE by imposing the notion of coherent beliefs. The coherent DSE strengthens the original DSE while still permitting deviations from PBE.

Before our game theoretic analysis, we derive a property of the Dempster–Shafer updating rule which, *per se*, is appealing from a decision theoretic perspective. Ambiguity designates a situation where probabilities are known only for some events. In signaling games, it is often taken for granted that a probability distribution on types is exogenously given. It is reasonable to assume that the Receiver incorporates the known probabilities into his ex-ante beliefs by revealing the Sender's types to be unambiguous, while messages might be ambiguous. In this paper, the assumption of unambiguous types is accommodated via Nehring's (Nehring, 1999) notion of "revealed unambiguous events". When some events are unambiguous (e.g., types) while other events are ambiguous (e.g., messages), it is interesting to ask under what updating rule the ex-ante non-ambiguity remains preserved under conditional preferences. It will be shown that this is the case under the pessimistic updating rule. More precisely, if the conditional Choquet preference is derived by the Dempster–Shafer updating rule, then the ex-ante unambiguous events continue to be perceived unambiguous after updating.[1]

This result has an important implication for Choquet preferences with respect to a capacity defined on a Cartesian product of two sets (e.g., a set of types and a set of messages). If the types are perceived ex-ante unambiguous and the conditioning event is an ambiguous message, then the conditional Choquet preference derived by the pessimistic updating rule coincides with the expected utility form. Both results hold true irrespective of whether the ex-ante capacity is convex or not (i.e., the Receiver exhibits ambiguity-aversion or not (see Schmeidler, 1989; Wakker, 2001)).

As a consequence, in signaling games under the pessimistic belief change, the only ambiguity the Receiver might perceive concerns messages at the ex-ante stage. After a message is observed, the ambiguity is "resolved" and the Receiver uses conditional probabilities for his inference about the Sender's private information. In this framework, we examine to what extent the DSE notion can explain behavior that is incompatible with the standard PBE.

Our game theoretic analysis begins with pooling DSE. It is shown that if the Sender's types are unambiguous, then pooling DSE has no descriptive power (i.e., no deviations from PBE are feasible). Under the pessimistic belief change, the Receiver's conditional beliefs on-the-equilibrium-path coincide with the prior probability distribution over the Sender's types and thus any pooling DSE can be explained by a "behaviorally equivalent" PBE.

However, in the context of separating behavior, DSE is more general than PBE. There exists a signaling game with separating equilibrium behavior that cannot be accommodated under the Bayesian paradigm (see Example 3). Although the Receiver exhibits conditional expected utility preference, the ex-ante ambiguity together with the pessimism inherent in the Dempster–Shafer rule hinders him from inferring (i.e., ascribing probability 1 to) the true type revealed by the Sender. In other words, the Receiver cannot infer the true type because he considers that there is another type than the true one who could also have sent the message observed. In our game, the Sender anticipates this and reveals her private information to ensure herself a certain payoff independent of the Receiver's response. Such behavior is inconsistent with the PBE, where separation without learning the true type is – due to Bayesian updating – infeasible.

The separating behavior just described has a particular feature. After receiving a message, the Receiver might regard a type as being "possible" which is not included in the support of his ex-ante equilibrium capacity.[2] Such belief change conflicts with the belief persistence axiom. If, however, one enforces the belief persistence axiom, the Receiver's preferences become severely constrained. More specifically, if the types are ex-ante unambiguous and the Receiver's beliefs comply with the belief persistence axiom in a separating DSE, then his ex-ante preferences must be subjective expected utility. Thus, the refined beliefs preclude any ambiguity and each separating DSE is PBE.

Since refinement based on the principle of belief persistence is too restrictive, we suggest an alternative criterion. As mentioned before, to obtain a separating DSE which is not PBE, the Receiver's conditional capacity has to assign a positive value to some types which were not included in the ex-ante support. Yet, the DSE definition itself does not impose any constraint on the support of conditional beliefs. In particular, types might be included in the conditional support for whom it is not "rational" to send the message observed. This "looseness" might give rise to Ryan's "troublesome" equilibrium behavior.

To eliminate such behavior, we strengthen the DSE definition by imposing an additional constraint on the Receiver's conditional beliefs. The condition requires the equilibrium beliefs to be coherent. The Receiver is said to hold *coherent* beliefs if for each type in the support of his conditional capacity there is a capacity under which the message observed is a best response of the Sender. The coherent DSE still admits deviations from separating PBE stemming from the Receiver's incapability of inferring the true type (such as in Example 3). However, the coherent DSE eliminates the Ryan-type behavior. More precisely, in signaling games in which revealing private information is the Sender's strictly dominant strategy, the only coherent beliefs are additive. In such games, the Receiver always infers the true types.

The paper is organized as follows: Section 2 recalls the capacity model together with the preference-based notion of unambiguous events. In Section 3, the Dempster–Shafer updating rule is defined and the decision-theoretical results are presented. In Section 4, the Dempster–Shafer equilibrium concept is presented. Section 5 investigates pooling and separating DSE behavior under the assumption of unambiguous types. In Section 6, Ryan's game is recalled and the notion of coherent Dempster–Shafer equilibrium is elaborated. Finally, Section 7 provides concluding remarks.

2. Choquet preferences and unambiguous events

Let S be a finite set of states. An event E is a subset of S and $\Sigma = 2^S$ the algebra of all events. For each $E \in \Sigma$, the complementary event $S \setminus E$ is denoted by E^c. A capacity $\nu : \Sigma \to \mathbb{R}$ is a monotone and normalized set function: (i) $\nu(\emptyset) = 0$, $\nu(S) = 1$ and (ii) $\nu(E) \leq \nu(F)$ for all $E \subseteq F$. A capacity ν on Σ is called convex, if it satisfies (iii) $\nu(E \cup F) + \nu(E \cap F) \geq \nu(E) + \nu(F)$ for all $E, F \in \Sigma$.[3] The results of this paper are not restricted to convex capacities, unless it is explicitly stated.

Let X be a set of consequences. An act $f : S \to X$ is a mapping from states to consequences. $\mathcal{A} = \{f \mid f : S \to X\}$ is the set of all

[1] This result holds true for the whole family of h-Bayesian updating rules, introduced by Gilboa and Schmeidler (1993), including the pessimistic updating rule as a special case.

[2] Note that the ex-ante support itself might be an ambiguous event and, thus, the states outside of the support do not need to be "impossible" in the sense of Savage-null events.

[3] Convex capacities express aversion towards ambiguity (see Schmeidler, 1989).

acts. For $f, g \in \mathcal{A}$, the act fEg assigns the outcome $f(s)$ to $s \in E$, and $g(s)$ to $s \in E^c$. A preference relation $\succcurlyeq$ on $\mathcal{A}$ is said to admit a Choquet expected utility representation if there exists a capacity ν on Σ and a utility function $u : X \to \mathbb{R}$ with respect to which any $f \in \mathcal{A}$ is evaluated via Choquet integration (see Choquet, 1954).

Definition 1. The Choquet integral of $f \in \mathcal{A}$ with respect to ν and u is

$$\int_S u(f)\, d\nu = \sum_{j=1}^{n} u(f(s_j))\Big[\nu(s_1,\ldots,s_j) - \nu(s_1,\ldots,s_{j-1})\Big],$$
$$\text{where } u(f(s_1)) \geq \cdots \geq u(f(s_n)) \text{ and } \nu(s_0) = 0.$$

The Choquet preferences have been axiomatized in different setups by Schmeidler (1989), Gilboa (1987), Wakker (1989), Sarin and Wakker (1992), Nakamura (1990), Chew and Karni (1994). Throughout the paper, $\succcurlyeq$ is a Choquet expected utility preference. If the Choquet integral is taken with respect to an additive capacity, then $\succcurlyeq$ is subjective expected utility (SEU) preference.

In choice problems under ambiguity, decision makers are informed about probabilities but only for some events. To incorporate the probabilistic piece of information into preferences, one needs to ensure that the events with known likelihoods are (subjectively) perceived as unambiguous. For this purpose, we adopt the notion of unambiguous events introduced by Nehring (1999).

Definition 2. An event $U \in \Sigma$ is unambiguous if, for every $A \in \Sigma$,

$$\nu(A) = \nu(A \cap U) + \nu(A \cap U^c), \tag{1}$$

otherwise U is ambiguous.

A capacity satisfying Condition (1) is said to be additively-separable across the unambiguous events.

Nehring's concept of unambiguous events has an axiomatic underpinning (see Sarin and Wakker, 1992; Dominiak and Lefort, 2011). An event $U \in \Sigma$ is revealed to be unambiguous by $\succcurlyeq$ if, and only if, $\succcurlyeq$ satisfies Savage's Sure-Thing Principle constrained to U and U^c. That is, for any $f, g, h, h' \in \mathcal{A}$ it is true that

$$fUh \succcurlyeq gUh \iff fUh' \succcurlyeq gUh', \tag{2}$$

and Condition (2) is also satisfied when U is everywhere replaced by U^c.[4]

Eichberger and Kelsey (1999) axiomatized an interesting class of capacities, called E-capacities, which make the inclusion of known probabilities tractable. Let p be a known probability distribution on S and let $\mathcal{P} = \{E_1, \ldots, E_I\}$ be a fixed partition of S. Assume that $\mathcal{P}$ is coarser than the finest partition of S.

Definition 3. For a given p on S, a partition $\mathcal{P}$ and a parameter $\rho \in [0, 1]$, the E-capacity $\nu_{p,\rho}(\cdot)$ on Σ is defined as follows: For any $A \in \Sigma$,

$$\nu_{p,\rho}(A) := \sum_{i=1}^{I}\big[\rho \cdot p(A \cap E_i) + (1-\rho)\cdot p(E_i)\cdot \beta_i(A)\big],$$
$$\text{where } \beta_i(A) = \begin{cases} 1 & \text{if } E_i \subseteq A, \\ 0 & \text{otherwise.} \end{cases}$$

A decision maker holding an E-capacity incorporates the probabilistic piece of information encoded by p, but only about the events in $\mathcal{P}$. That is to say, the decision maker is confident that p truthfully describes the likelihoods of the events in $\mathcal{P}$, but otherwise distorts probabilities by a degree of confidence ρ.

As the following lemma shows, all events in $\mathcal{P}$ are unambiguous.

Lemma 1. *Fix a partition $\mathcal{P}$ of S and a probability distribution p on S. Let $\nu_{p,\rho}(\cdot)$ be an E-capacity based on p and $\mathcal{P}$ with a degree of confidence $\rho \in [0, 1]$. Then, every $E \in \mathcal{P}$ is unambiguous.*

It is worth mentioning that for the case of convex capacities, the notion of unambiguous events coincides with the standard additivity definition. That is, if ν is convex and $\nu(U) + \nu(U^c) = 1$ for some $U \in \Sigma$, then U is unambiguous (see Nehring, 1999). However, when ν is non-convex, the additivity condition does not need to imply that U is unambiguous. This point is illustrated below.

Example 1. Let $S = \{R, B, Y\}$. Consider the following capacity ν on 2^S:

$$\nu(R) = \frac{1}{3}, \qquad \nu(B \cup Y) = \frac{2}{3},$$
$$\nu(B) = \frac{1}{6}, \qquad \nu(R \cup B) = 1,$$
$$\nu(Y) = \frac{1}{6}, \qquad \nu(R \cup Y) = 1.$$

The capacity is not convex and is additive on R and R^c. Yet, R is ambiguous,

$$1 = \nu(R \cup B) \neq \nu(R) + \nu(B) = \tfrac{1}{2}.$$

3. The *h*-Bayesian updating rules and ambiguous events

In dynamic choice situations, new information is observed in the form of an event E. The new piece of information is embedded into the decision making process by updating the unconditional preference $\succcurlyeq$. In this section, the conditional preference, denoted by $\succcurlyeq_E$, is derived according to an h-Bayesian updating rule introduced by Gilboa and Schmeidler (1993).

Definition 4. An h-Bayesian updating rule is defined as follows: there exists $D \in \Sigma$ such that for all $f, g \in \mathcal{A}$ and all $E \in \Sigma$,

$$f \succcurlyeq_E g \iff fEh \succcurlyeq gEh, \tag{3}$$

where $h = \bar{x}D\underline{x}$ with $\bar{x}$ and $\underline{x}$ denoting the best and worst consequence in X.[5]

Under the h-Bayesian updating rules, the conditional preference $\succcurlyeq_E$ admits Choquet expected utility representation with respect to a conditional capacity ν_E and the unconditional utility function u (see Gilboa and Schmeidler, 1993).

We remark that the h-Bayesian updating rules respect consequentialism, a fundamental property of conditional preferences introduced by Hammond (1988), but not dynamic consistency. It is well known that one of these two properties must be relaxed when modeling ambiguity-sensitive behavior (see Ghirardato, 2002). In a recent experimental study, Dominiak et al. (2012) provide an evidence confirming the view that subjects behave more in line with consequentialism than dynamic consistency.

Our first result derives an appealing property of the family of h-Bayesian updating rules. It is shown that conditional Choquet

4 There are other notions of unambiguous events proposed by Epstein and Zhang (2001), Zhang (2002), and further explored by Kopylov (2007). Roughly, their notions weaken the Sure-Thing Principle by assuming *probabilistic sophistication* in the sense of Machina and Schmeidler (1992) as benchmark to evaluate unambiguous acts; i.e., acts measurable with respect to unambiguous events.

5 It is assumed that such consequences exist.

preferences derived by the h-Bayesian updating rules preserve the additive-separability condition (1). In other words, events that are ex-ante unambiguous remain unambiguous after updating.[6]

Theorem 1. *Let $\succcurlyeq$ be an unconditional Choquet preference relation, $U \in \Sigma$ an unambiguous event revealed by $\succcurlyeq$, and $E \in \Sigma$ a conditioning event. If the conditional preferences $\succcurlyeq_E$ are derived by an h-Bayesian updating rule, then U remains unambiguous after updating. That is, for all acts $f, g, h, h' \in \mathcal{A}$:*

$$fUh \succcurlyeq_E gUh \iff fUh' \succcurlyeq_E gUh'. \tag{4}$$

Note that Condition (4) is tantamount to saying that the conditional capacity ν_E maintains the additive-separability condition across the unambiguous events. That is, for all $A \in \Sigma$, it is true that

$$\nu_E(A) = \nu_E(A \cap U) + \nu_E(A \cap U^c). \tag{5}$$

From now on, we will explore the implications of the above property under the special h-Bayesian updating rule where h is the constant act assigning the best consequence $\bar{x}$ to all states (i.e., $D = S$).[7] In this case, the conditional capacity ν_E is revised according to the Dempster–Shafer rule introduced by Dempster (1968) and Shafer (1976) (see Gilboa and Schmeidler, 1993, p. 42).

Definition 5. Let ν be a capacity and $E \in \Sigma$ a conditioning event. The Dempster–Shafer updating rule is defined as follows: For every $A \in \Sigma$,

$$\nu_E(A) = \frac{\nu\big((A \cap E) \cup E^c\big) - \nu(E^c)}{1 - \nu(E^c)},$$

whenever $\nu(E^c) < 1$.

The Dempster–Shafer rule reflects a pessimistic belief change. When E is observed, the decision maker believes that the best consequence belongs to the complementary event E^c, and so it is impossible to occur. The conditioning events are always regarded as "not-good-news".

The next example illustrates the meaning of Theorem 1 in the context of a dynamic version of the 3-color experiment of Ellsberg (1961).

Example 2. Consider an Ellsberg-urn containing 90 balls; 30 balls are Red and 60 balls are either Blue or Yellow. There is no further information on the composition. Let $S = \{R, B, Y\}$ be the state space and ν the capacity on 2^S:

$$\nu(R) = \frac{1}{3}, \qquad \nu(B \cup Y) = \frac{2}{3},$$
$$\nu(B) = \frac{1}{6}, \qquad \nu(R \cup B) = \frac{1}{2},$$
$$\nu(Y) = \frac{1}{6}, \qquad \nu(R \cup Y) = \frac{1}{2}.$$

A decision maker incorporates the probabilistic piece of information and perceives R and R^c to be unambiguous events. One ball is randomly drawn and the decision maker is informed that the color is not Yellow, i.e., $E = \{R, B\}$. Given this information, the Dempster–Shafer rule delivers

$$\nu_E(R) = \frac{2}{5} \quad \text{and} \quad \nu_E(B) = \frac{3}{5}.$$

Since R remains unambiguous after updating and there are only two states when conditioning on E, event B becomes unambiguous and the decision maker does not perceive ambiguity.

This property of the pessimistic belief change becomes even more meaningful when capacities are defined over a product space of two finite sets $\mathcal{T}$ and $\mathcal{M}$. Consider a capacity ν defined on $2^{\mathcal{T}\times\mathcal{M}}$. Suppose that the partition $\mathcal{P}_{\mathcal{T}} = \big\{\{t\} \times \mathcal{M} \mid t \in \mathcal{T}\big\}$ is unambiguous while $\mathcal{P}_{\mathcal{M}} = \big\{\mathcal{T} \times \{m\} \mid m \in \mathcal{M}\big\}$ might be ambiguous with respect to ν. If the conditioning event E is an element of $\mathcal{P}_{\mathcal{M}}$, the conditional preference $\succcurlyeq_E$ is subjective expected utility.

Lemma 2. *Let $\succcurlyeq$ be an unconditional Choquet preference with respect to a utility function u and a capacity ν on $2^{\mathcal{T}\times\mathcal{M}}$. Suppose that the elements in partition $\mathcal{P}_{\mathcal{T}}$ are unambiguous and that for every $E \in \mathcal{P}_{\mathcal{M}}$, the conditional preference $\succcurlyeq_E$ is derived by the Dempster–Shafer updating rule and well defined. Then, $\succcurlyeq_E$ is SEU preference.*

The result seems intuitive. In the product space where one coordinate is unambiguous while the other is not, a state consists of two components: the ambiguous and unambiguous state. After the uncertainty governing the ambiguous state is resolved, the true state only depends upon the component that was ex-ante unambiguous. The Dempster–Shafer rule preserves the ex-ante non-ambiguity and the decision maker behaves as an expected utility maximizer with respect to an updated probability distribution on $\mathcal{T}$.

In the following sections, we explore the implications of the results above in the context of signaling games in which messages are perceived ambiguous and the pessimistic Dempster–Shafer updating rule is used for updating beliefs.[8]

4. Dempster–Shafer equilibrium in signaling games

The class of signaling games studied here is described as follows. There are two players called "the Sender" (S) and "the Receiver" (R). In the ex-ante stage, Nature draws a type for the Sender from the set of types $\mathcal{T} = \{t_j\}_{j=1}^J$ according to a probability distribution p defined on $\mathcal{T}$. The Sender learns her type and then chooses a message from $\mathcal{M} = \{m_k\}_{k=1}^K$, the set of messages. In the interim stage, the Receiver observes the message, but not the type, and selects a response from $\mathcal{R} = \{r_l\}_{l=1}^L$, the set of responses, and the game ends. Final payoffs are given by $u^i : \mathcal{T} \times \mathcal{M} \times \mathcal{R} \to \mathbb{R}$ for $i \in \{S, R\}$. We assume $J, K, N \geq 2$ and denote this class of signaling games by Γ.

The players hold beliefs about the opponent's behavior. The Sender's beliefs are about the Receiver's responses that will be chosen after observing message m. They are represented by a family ν^S of capacities $\{\nu^S_{(m)}\}_{m\in\mathcal{M}}$, where each $\nu^S_{(m)}$ is defined on $2^{\mathcal{R}}$. The Receiver's capacity ν^R is defined on Σ^R, the set of all subsets of $\mathcal{T} \times \mathcal{M}$. It represents the Receiver's ex-ante joint belief about the Sender's strategic choice of messages and her possible types.

In the interim stage, the Receiver observes the message sent by the Sender and revises his beliefs according to the Dempster–Shafer rule. His conditional beliefs are represented by the family of updated capacities $\{\nu^R_m\}_{m\in\mathcal{M}}$.[9]

For notational convenience, we denote by $T_j := \{t_j\} \times \mathcal{M}$ and $M_k := \mathcal{T} \times \{m_k\}$ the marginal events of the product space $\mathcal{T} \times \mathcal{M}$. We follow the previous notations: $\mathcal{P}_{\mathcal{T}} := \{T_j\}_{j=1}^J$ and $\mathcal{P}_{\mathcal{M}} := \{M_k\}_{k=1}^K$ to denote the partitions.

6 It merits emphasis that another updating rule for non-additive beliefs, the Full-Bayesian updating rule, suggested by Walley (1991) and Jaffray (1992), does not respect this property.

7 Another special case refers to the h-Bayesian updating rule where h assigns the worst consequence $\underline{x}$ to all states (i.e., $D = \emptyset$). In this case, the conditional capacity ν_E is revised according to Bayes' rule.

8 Exploration of the effects of the other h-Bayesian rules on the equilibrium behavior in signaling games is beyond the scope of this paper and the subject of ongoing research.

9 For convenience, we write ν^R_m to denote the conditional capacity on $M := \mathcal{T} \times \{m\}$.

In games under incomplete information, it is taken for granted that the probability distribution p is publicly available to the players. To incorporate the probabilistic piece of information into the Receiver's ex-ante beliefs, we follow Eichberger and Kelsey (2004) and assume that ν^R agrees with p on $\mathcal{T}$.

Assumption 1. The Receiver's capacity ν^R agrees with p on $\mathcal{T}$. That is,

$$\nu^R(T_j) = p(t_j) \quad \text{for all } j = 1, \dots, J. \tag{6}$$

As mentioned before, a (non-convex) capacity which agrees with p on $\mathcal{T}$ does not necessarily imply that each type is unambiguous. For this to be true, the additive-separability condition (1) of Definition 2 is further required.[10]

Assumption 2. The Sender's types are unambiguous. That is, the Receiver's capacity ν^R satisfies the following condition: For all $A \subseteq \mathcal{T} \times \mathcal{M}$ and all $j = 1, \dots, J$,

$$\nu^R(A) = \nu^R(A \cap T_j) + \nu^R(A \cap T_j^c). \tag{7}$$

The Dempster–Shafer equilibrium (DSE) constitutes an equilibrium in beliefs.[11] The equilibrium definition relies on a support notion. Following Eichberger and Kelsey (2004), the definition of Dow and Werlang (1994) is adopted.[12] A Dow–Werlang (DW)-support is a smallest event whose complement has the capacity value zero.

Definition 6. A *DW*-support of a capacity ν, denoted by $supp(\nu)$, is an event $D \subseteq S$ such that $\nu(D^c) = 0$ and $\nu(F^c) > 0$, for any $F \subset D$.

The definition of DSE consists of three components. The first two conditions require the consistency of the actual behavior with the players' beliefs about the opponents' actions. That is, any action which belongs to the support of a player must be a best response of the opponent. The last condition requires the Receiver to follow the Dempster–Shafer updating rule whenever possible.

Definition 7. A Dempster–Shafer equilibrium (DSE) consists of families of beliefs $\left[\nu^S, \nu^R, \{\nu_m^R\}_{m\in\mathcal{M}}\right]$ for which there exist associated supports satisfying

(i) $(t^*, m^*) \in \text{supp}(\nu^R)$

$$\Rightarrow \; m^* \in \arg\max_{m\in\mathcal{M}} \int_{\mathcal{R}} u^S(m, r, t^*) d\nu^S_{(m)},$$

(ii) $r^*(m) \in \text{supp}(\nu^S_{(m)})$

$$\Rightarrow \; r^*(m) \in \arg\max_{r\in\mathcal{R}} \int_{\mathcal{T}} u^R(m, r, t) d\nu^R_m \quad \forall m \in \mathcal{M},$$

(iii) ν_m^R is derived by the

Dempster–Shafer updating rule whenever possible.

If all capacities are additive, the definition of DSE coincides with the definition of perfect Bayesian equilibrium (PBE) in beliefs.[13] The existence of a DSE is guaranteed (see Eichberger and Kelsey, 2004).

There is one remark in order. If ν_m^R is not well defined after observing a message m, the conditional beliefs are chosen arbitrarily in analogy to PBE. However, in our setup, the conditional capacities are always additive whenever they are well defined (see Lemma 2). Therefore, we assume that the arbitrarily chosen conditionals are also additive.[14]

In signaling games, two sorts of equilibria are of special interest, the separating and pooling equilibrium in pure strategies. Following the convention, in a separating DSE, each type of the Sender sends a different message; in a pooling DSE, all types of the Sender send the same message.

Definition 8. Let $\left[\nu^S, \nu^R, \{\nu_m^R\}_{m\in\mathcal{M}}\right]$ be a DSE and $supp(\nu^R)$ be an ex-ante support associated with the equilibrium capacity ν^R satisfying condition (i) of Definition 7. Then, the DSE is called

(i) separating if, $\psi_t(\nu^R) \cap \psi_{t'}(\nu^R) = \emptyset$ for all $t, t' \in \mathcal{T}$, and
(ii) pooling if, $\psi_t(\nu^R) = \psi_{t'}(\nu^R)$ for all $t, t' \in \mathcal{T}$,

where $\psi_t(\nu^R) = \left\{m \in \mathcal{M} \mid (t, m) \in supp(\nu^R)\right\}$ and $|\psi_t(\nu^R)| = 1$ for any $t \in \mathcal{T}$.

For a given DSE, $\left[\nu^S, \nu^R, \{\nu_m^R\}_{m\in\mathcal{M}}\right]$, denote by

$$\mathcal{B}(DSE) = \left\{\{supp(\nu^S_{(m)})\}_{m\in\mathcal{M}}, supp(\nu^R), \{supp(\nu^R_m)\}_{m\in\mathcal{M}}\right\}$$

the equilibrium actions supported by the equilibrium beliefs. Two equilibria, DSE^1 and DSE^2, are called *behaviorally equivalent* if they support the identical equilibrium actions; i.e., $\mathcal{B}(DSE^1) = \mathcal{B}(DSE^2)$.

Since we are interested in equilibrium behavior caused solely by the pessimistic belief change, we restrict our attention to equilibria in which the Sender holds additive beliefs.

5. Equilibrium behavior under unambiguous types

In signaling games in which the Sender's types are assumed to be unambiguous, the Receiver can only display ambiguity-sensitive preferences in the ex-ante stage. After a message is observed, however, the Receiver no longer perceives ambiguity and uses a conditional probability measure to make an inference about the Sender's private information (see Lemma 2). Given this "least" departure from the Bayesian framework, we will examine whether ex-ante ambiguity is sufficient for the existence of a DSE that is incompatible with the standard PBE notion.

First of all, it should be stressed that the assumption of unambiguous types substantially constrains pooling DSE by eliminating any deviation from PBE. The reason is the following. In a pooling DSE, the Receiver's conditional beliefs on-the-equilibrium-path coincide with the prior distribution over the Sender's types[15] and beliefs off-the-equilibrium path are additive anyway. Thus, one can "substitute" the Receiver's ex-ante capacity with the prior probability distribution over types and derive his conditional beliefs on and off-the-equilibrium-path as dictated by the Dempster–Shafer updating rule. The "substituted" probability measures must support the same pooling behavior under PBE.[16]

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10 In Eichberger and Kelsey (2004), capacities are assumed to be convex. Hence, our setup is more general.

11 The DSE is an extension of the notion of "equilibrium in beliefs" in static games introduced by Dow and Werlang (1994); Eichberger and Kelsey (2000); and further studied by Marinacci (2000) and Haller (2000). The notion of equilibrium in beliefs considers only pure strategies. There exists another approach in the literature which explicitly allows for mixed strategies, including Lo (1996), Klibanoff (1996), Bade (2011) and Lehrer (2012) for the static games; Kajii and Ui (2005) and Azrieli and Teper (2011) for dynamic games under incomplete information. Most recently, Klibanoff et al. (2015) extend the notions of sequential equilibrium and PBE to the family of smooth ambiguity preferences of Klibanoff et al. (2005).

12 A DW-support always exists, but it may not be unique. There is another support notion introduced by Marinacci (2000). It is a set of states which have a strictly positive capacity value. The Marinacci-support is unique, provided it exists. It is included in any DW-support (see Eichberger and Kelsey, 2014, Lemma 1). Our results carry over to setups under the notion of Marinacci-support. For a more comprehensive discussion on the various supports, see Ryan (2002b), Eichberger and Kelsey (2014), or Dominiak and Eichberger (2016).

13 Note that the Dempster–Shafer rule coincides with Bayes' rule if a capacity is additive.

14 This assumption is relevant only when studying pooling DSE.

15 See Lemma 3 in Appendix D.

16 However, the Receiver's conditional beliefs might be non-additive when the types are ambiguous. Appendix B provides a pooling DSE which cannot be supported by any PBE.

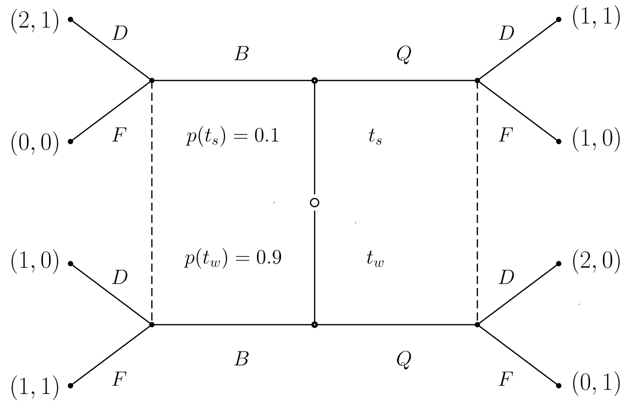

Fig. 1. Separating equilibrium under ambiguity.

Proposition 1. *Consider a signaling game in Γ. Assume that the Sender's types are unambiguous. Then, for any pooling DSE, provided it exists, there exists a behaviorally equivalent* PBE *(i.e.,* $\mathcal{B}(\text{DSE}) = \mathcal{B}(\text{PBE})$*).*

However, for separating DSE, the assumption of unambiguous types is less restrictive. That is to say, the DSE notion can accommodate separating behavior that cannot be captured by PBE, even though types are unambiguous. Below, an example of such behavior is illustrated.

Example 3. Consider the signaling game depicted in Fig. 1, where t_s stands for the strong type and t_w is the weak type. It can be verified that the game has neither pooling nor separating (pure-strategy) PBE.[17] However, when the Receiver's beliefs are non-additive, the separating behavior where the strong type sends Q and weak type sends B is supported by a DSE. The family of beliefs constituting the separating DSE is defined in Tables 1–3. For given α_1 and α_2, the Receiver's conditional Choquet expected utility is

$$\int_{\mathcal{T}} u^R(m, r, t)\, d\nu_m^R = \begin{cases} \frac{0.1}{1-\alpha_2} & \text{if } (m, r) = (Q, D), \\ \frac{0.9-\alpha_2}{1-\alpha_2} & \text{if } (m, r) = (Q, F), \\ \frac{0.1-\alpha_1}{1-\alpha_1} & \text{if } (m, r) = (B, D), \\ \frac{0.9}{1-\alpha_1} & \text{if } (m, r) = (B, F). \end{cases}$$

The types are unambiguous with respect to the Receiver's ex-ante capacity. The messages are ambiguous. Since the capacity is additively-separable across the types, the states in the ex-ante support and the support are ambiguous (i.e., $\nu^R(supp(\nu^R)) = \alpha_1 + \alpha_2 < 1$). In other words, there is ambiguity about the separating behavior. Because of the ex-ante ambiguity and the pessimistic updating, the Receiver cannot infer the true type from the message observed. Although the Receiver uses a conditional probability for his inferences about types, he does not assign probability 1 to the type who sent the message on-the-equilibrium-path.[18] the Receiver always infers that the weak type is more likely and thus it is optimal for him to respond with F regardless of the message he observes. This, in fact, induces the Sender to separate. That is, by anticipating that the Receiver is incapable of learning her private information, the Sender will secure the certain payoff 1 rather than exposing herself to "strategic" uncertainty.

Table 1
Receiver's ex-ante beliefs.

E	$\nu^R(E)$
$\{(t_s, Q)\}$	α_1
$\{(t_s, B)\}$	0
$\{(t_w, Q)\}$	0
$\{(t_w, B)\}$	α_2
$\{(t_s, B), (t_s, Q)\}$	0.1
$\{(t_w, B), (t_w, Q)\}$	0.9
$\{(t_s, Q), (t_w, Q)\}$	α_1
$\{(t_s, B), (t_w, B)\}$	α_2
$\{(t_s, B), (t_w, Q)\}$	0
$\{(t_s, Q), (t_w, B)\}$	$\alpha_1 + \alpha_2$
$\{(t_s, B), (t_s, Q), (t_w, B)\}$	$0.1 + \alpha_2$
$\{(t_s, B), (t_s, Q), (t_w, Q)\}$	0.1
$\{(t_w, B), (t_w, Q), (t_s, Q)\}$	$0.9 + \alpha_1$
$\{(t_w, B), (t_w, Q), (t_s, B)\}$	0.9

$0 < \alpha_1 \leq 0.1$ and $0 < \alpha_2 < 0.8$
$supp(\nu^R) = \{(t_s, Q), (t_w, B)\}$.

Table 2
Receiver's conditional beliefs.

m	E	$\nu_m^R(E)$
Q	T_s	$\frac{0.1}{1-\alpha_2}$
	T_w	$\frac{0.9-\alpha_2}{1-\alpha_2}$
B	T_s	$\frac{0.1-\alpha_1}{1-\alpha_1}$
	T_w	$\frac{0.9}{1-\alpha_1}$

$supp(\nu_B^R) = \{(t_s, B), (t_w, B)\}$ if $\alpha_1 < 1$
$supp(\nu_B^R) = \{(t_w, B)\}$ if $\alpha_1 = 1$
$supp(\nu_Q^R) = \{(t_s, Q), (t_w, Q)\}$.

Table 3
Sender's beliefs.

m	E	$\nu_{(m)}^S(E)$
Q	$\{D\}$	0
	$\{F\}$	1
B	$\{D\}$	0
	$\{F\}$	1

The driving force behind such deviation from PBE is the ex-ante ambiguity together with the pessimistic attitude of the Dempster–Shafer updating rule. Notice that prior to observing a message, the support of the Receiver's ex-ante capacity contains the state that the strong type sends message Q but does not include the state that the weak type sends Q (see Table 1). However, the ex-ante support is an ambiguous event and so is the state that the weak type sends Q, even though it has the capacity value zero. In other words, the Receiver does not entirely preclude the possibility that the weak type sends Q. After observing Q, the Receiver's conditional capacity, which is a probability distribution, attaches positive values to *both* types. This is an essential difference to the separating behavior supported by PBE. In the DSE, the pessimistic belief change hinders the Receiver from inferring the true type, although the Sender reveals her private information by sending a different message for each type. This behavior is impossible under the Bayesian paradigm where, in any separating PBE, the Receiver learns the Sender's private information by ascribing probability 1 to the type who sent the message on-the-equilibrium-path.

However, Ryan (2002a) questioned the notion of DSE. He pointed out that the Receiver's beliefs on-the-equilibrium-path may conflict with the so-called belief persistence axiom. Broadly speaking, the belief persistence axiom is a qualitative requirement

[17] Notice that separating PBE does not exist because the Sender has an incentive to deviate once she anticipates that the Receiver will learn the Sender's true type from a message. For instance, the separation where the strong type sends Q, while the weak types sends B, cannot constitute a PBE. Once the Receiver learns that Q has been sent by the strong type, he optimally responds by playing D. Knowing that D is played after Q, the weak type should profitably deviate by sending Q instead of B. The only PBE is in mixed strategies; $\left[\left(Q, \left(\frac{8}{9}B + \frac{1}{9}Q\right)\right), \left(F, \left(\frac{1}{2}D + \frac{1}{2}F\right)\right)\right]$. The strong type sends Q while the weak type mixes between B and Q; the Receiver responds with F after observing message B, and mixes between D and F after observing Q.

[18] Only when $\alpha_1 = 0.1$, the Receiver infers that B has been sent by the weak type.

for updated beliefs to reflect the ex-ante beliefs as accurately as possible (see Battigalli and Bonanno, 1997).[19]

Definition 9. Let ν be a capacity on Σ and ν_E be a conditional capacity on an event E. Fix $supp(\nu)$, a support of ν and $supp(\nu_E)$, a support of ν_E. The capacities ν and ν_E are said to respect the belief persistence axiom on E if,

$$supp(\nu) \cap E \neq \emptyset \Longrightarrow supp(\nu_E) = supp(\nu) \cap E. \tag{8}$$

In Example 3, the Receiver's conditional capacity assigns a strictly positive value to the weak type after observing Q, although the state that the weak type sends Q is outside of the ex-ante support. To put it differently, at a separating DSE, the support of the Receiver's conditional capacity may "expand", relative to the ex-ante support, by adding the states that initially had the capacity value zero. Ryan advocated to refine a DSE by considering only beliefs that comply with belief persistence. However, if the belief persistence axiom is exogenously imposed on equilibrium beliefs, then a remarkable result follows.

Proposition 2. *Consider a signaling game in Γ with $|\mathcal{M}| = |\mathcal{T}| \geq 2$. Assume that the Sender's types are unambiguous. At a separating* DSE*, provided it exists, if the Receiver's equilibrium beliefs satisfy the belief persistence axiom on every $m \in \mathcal{M}$ and its complement, then his ex-ante preference is* SEU*.*

Under the joint assumptions of unambiguous types and belief persistence, the separating equilibrium precludes any ambiguity.

Notice, however, that the belief persistence axiom is assumed to hold true on the complement of every message. This may be seen as an unnecessarily strong requirement since such events will never be observed in signaling games. Yet, if the belief persistence axiom is satisfied only on messages, the previous result remains valid whenever the Receiver is ambiguity averse.[20]

Proposition 3. *Consider a signaling game in Γ with $|\mathcal{M}| = |\mathcal{T}| \geq 2$. Assume that the Sender's types are unambiguous. At a separating* DSE*, provided it exists, if the Receiver's equilibrium capacity is convex and it satisfies the belief persistence axiom on every $m \in \mathcal{M}$, then his ex-ante preference is* SEU*.*[21]

To illustrate a violation of the belief persistence axiom, Ryan provided the signaling game depicted in Fig. 2. The equilibrium beliefs for his game are presented in Appendix A (Ryan's DSE refers to Case 1).[22] In his analysis, the Receiver's capacity is assumed to be an E-capacity. Since E-capacities are convex[23] and they reveal types to be unambiguous (by Lemma 1), Ryan's result (2002a, Proposition 4.1) is a corollary of Proposition 3.

Corollary 1. *In any separating* DSE *of Ryan's game which respects the belief persistence axiom, the Receiver displays ex-ante* SEU *preferences.*

These findings detect some limitations of separating DSE under the belief persistence axiom. Roughly speaking, the axiom forces the Receiver to exhibit ex-ante SEU preferences. Thus, the pessimism inherent in the Dempster–Shafer updating rule vanishes and the strategic behavior will not differ from that of PBE. As a consequence, deviations from PBE, as discussed in Example 3, cannot be modeled via DSE unless the belief persistence axiom is abandoned.

6. Coherent Dempster–Shafer equilibrium

The main rationale for Ryan's (2002a) refinement concept was his observation that the DSE might support, what he calls, "implausible" behavior. However, if equilibrium beliefs are restricted by the belief persistence axiom then, deviations from the PBE become infeasible, and the DSE notion loses its descriptive power. In this section, an alternative refinement is presented. Our refinement is less stringent; while it successfully eliminates Ryan's behavior, it is flexible enough to maintain the separating DSE of Example 3.

Let us briefly recall Ryan's argument. His game is depicted in Fig. 2. Let m and m^o be two distinct messages. The message m is said to be strictly dominated by m^o for a type t^o, if the following holds true

$$\min_{r \in \mathcal{R}} u^S(m^o, r, t^o) > \max_{r \in \mathcal{R}} u^S(m, r, t^o). \tag{9}$$

When all the messages $m \neq m^o$ are strictly dominated by m^o for the type t^o, then m^o is called the strictly dominant message for t^o. Notice that message L (resp. R) is the strictly dominant message for t_1 (resp. t_2). Therefore, Ryan (2002a, p. 170) argues that it "*seems reasonable that player 2 ought to choose U in any sensible analysis of this game*". However, since the Receiver might perceive ambiguity about messages, it is possible to construct a DSE in which the Sender plays her strictly dominant strategy while the Receiver always plays D giving him the payoff 0. A family of capacities supporting such DSE is presented in Appendix A, Case 1. Ryan views such DSE as "troublesome" or "implausible". It should be noted, however, that this "implausible" behavior is not caused, *per se*, by the violation of the belief persistence axiom. There exists a family of capacities supporting the separating DSE that is behaviorally equivalent to the unique PBE in which the Receiver's best response is U (see Appendix A, Case 2). Nevertheless, such equilibrium beliefs also violate the belief persistence axiom.[24]

In the separating DSE of Ryan's game, it is again the pessimism of the updating rule that prevents the Receiver from inferring the true type. At the interim stage, the Receiver regards both types of t_1 and t_2 as possible (i.e., both types are in the support of his conditional capacities). After observing L, the Receiver optimally responds with D by believing that t_2 could also have sent the message observed. However, for type t_2, sending L will never be rational. There is no additive belief (and in fact, no capacity) over the Receiver's responses (after observing L) with respect to which sending L is optimal for t_2. The same argument applies to type t_1 who sends message R. Therefore, the Receiver's beliefs are inconsistent with a standard rationality assumption that the Sender will never play a strictly dominated strategy.

Motivated by this observation, we suggest a strengthening of DSE by requiring the Receiver's beliefs to be *coherent* with the assumption that the Sender is *Choquet rational*, introduced by

19 The belief persistence axiom is one of the principles postulated in the theory of belief change in the tradition of Alchourron et al. (1985). Stalnaker (1998) provides a concise exposition on the AGM theory. Ryan (2001) derives the conditions under which the Dempster–Shafer updating rule satisfies all the AGM-axioms.

20 Example 4 in Appendix C illustrates that the non-convex capacities respecting the belief persistence axiom on every message in a separating DSE do not need to be additive.

21 The assumption $|\mathcal{M}| = |\mathcal{T}|$ is necessary (see Example 5 in Appendix C).

22 Ryan's game and his argument are further explored in Section 6.

23 See Lemma 2.1 in Eichberger and Kelsey (1999).

24 Since the equilibrium capacity is an E-capacity, it is worth to note that ρ, the degree of confidence in the probability distribution supporting the unique separating PBE, determines whether the Receiver's best reply is D or U. His best reply is D if, $0 < \rho \leq \frac{2}{3}$ and U if, $\frac{2}{3} \leq \rho \leq 1$ (see Cases 1 and 2 in Appendix A).

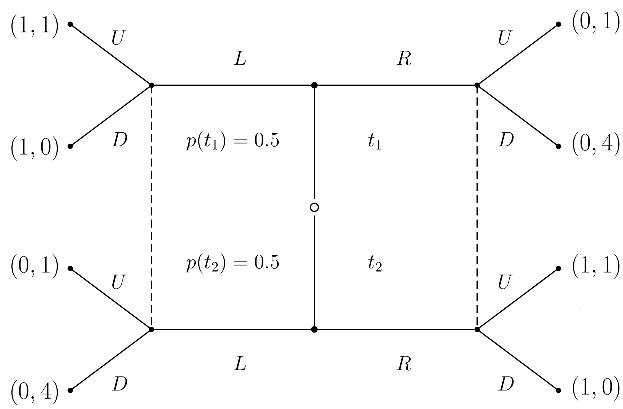

Fig. 2. Signaling game of Ryan.

Ghirardato and Le Breton (2000), in the sense that she never sends a message that cannot be rationalized by any capacity.[25]

Definition 10. Fix a signaling game in Γ. A collection $\left\{\nu^R, \{\nu^R_m\}_{m\in\mathcal{M}}\right\}$ of the Receiver's beliefs is said to be *coherent*, if for each $m \in \mathcal{M}$ and $t \in supp(\nu^R_m)$, there exists a capacity ν^S on $2^{\mathcal{R}}$ such that

$$m \in B(t) = \arg\max_{m' \in \mathcal{M}} \int_{\mathcal{R}} u^S(m', r, t)\, \mathrm{d}\nu^S. \tag{10}$$

Definition 11. A Coherent Dempster–Shafer Equilibrium (CDSE) is a DSE $\left[\nu^S, \nu^R, \{\nu^R_m\}_{m\in\mathcal{M}}\right]$ in which the Receiver's beliefs are coherent.

It is important to realize that the separating DSE of Example 3 is very different from the DSE of Ryan's game. Although the Receiver believes that both types could have sent the message on-the-equilibrium-path (thus violating the belief persistence axiom), for each type off-the-equilibrium-path there exists a capacity (even an additive one) which rationalizes sending the message that the Receiver factually observes. The separating DSE of Example 3 is coherent while the DSE of Ryan's game is not.

The notion of coherent beliefs constrains the Receiver's equilibrium beliefs as the belief persistence does. However, there is a substantial difference. The restriction imposed by the coherence notion is weaker and directly related to the players' payoff structure. Consider a separating DSE in which there is a type t^o with the strictly dominant message m^o on-the-equilibrium-path. If the Receiver's beliefs are coherent then the state (t^o, m^o) is ex-ante unambiguous. Nonetheless, the message m^o might still be ambiguous.

Proposition 4. *Consider a signaling game in Γ for which there exists (t^o, m^o) such that m^o is the strictly dominant message for the Sender of type t^o. Assume that the Receiver's capacity is convex and the Sender's types are unambiguous. If $(t^o, m^o) \in supp(\nu^R)$ at a* CDSE, *$\left[\nu^S, \nu^R, \{\nu^R_m\}_{m\in\mathcal{M}}\right]$, then (t^o, m^o) is unambiguous with respect to ν^R.*

Thus, the notion of coherent beliefs implies that the Receiver learns the type for whom sending the message observed is the strictly dominant strategy.

This finding implies that in signaling games in which each type has a strictly dominant message, the Receiver with convex capacity must exhibit an ex-ante SEU preference in any separating CDSE.

[25] Note that the Sender's (additive) beliefs are always coherent due to the condition (ii) of the definition of DSE.

Corollary 2. *Let $\left[\nu^S, \nu^R, \{\nu^R_m\}_{m\in\mathcal{M}}\right]$ be a CDSE for a signaling game in Γ. Suppose that for every pair $(t, m) \in supp(\nu^R)$, m is the strictly dominant message for the Sender type t. If ν^R is convex and the Sender's types are unambiguous, then the Receiver's ex-ante preference is* SEU.

In Ryan's game, the restrictions imposed by the coherent beliefs lead to the same conclusion as previously derived under the belief persistence axiom. Since each type has the strictly dominant message, the Receiver must have SEU preferences at any coherent DSE, which is in fact the unique separating PBE of Ryan's game.

7. Concluding remarks

In signaling games the Receiver might update his ex-ante beliefs in a pessimistic manner when incorporating the arrival of an ambiguous message. The Dempster–Shafer equilibrium is a solution concept that captures such updating behavior, and thus, generalizes the perfect Bayesian equilibrium.

In this study, we conducted a sort of litmus test for the DSE concept. Following Aumann (1985), "[...] a solution concept should be judged more by what it does than by what it is". In line with such motivation, we examined what the DSE can accomplish when compared to the standard PBE notion.

Our main findings unfolded the tension between the descriptive power of the DSE and two assumptions imposed on equilibrium beliefs: the assumption of unambiguous types and the belief persistence axiom. Under the former assumption, the DSE notion admits deviations from separating but not from pooling PBE. However, when both assumptions are simultaneously enforced, ambiguity cannot exist at all and every separating DSE is PBE. These results disclose the descriptive limitation of DSE under both assumptions. Yet, the latter assumption is controversial.

Why should not it be admissible to relax the belief persistence axiom? Recall that one part of the belief persistence axiom requires that, if the Receiver regards a state as "impossible", then he should continue to regard it as "impossible" after a new piece of information is observed. The term "impossibility" is an essential component but not clearly specified. When beliefs are additive, the "impossible" states are those outside of the support; these states are *de facto* unambiguously "impossible" (i.e., Savage-null). The belief persistence axiom, therefore, is naturally satisfied. However, when beliefs are non-additive, not all states outside of a support are necessarily Savage-null. Bearing in mind that beliefs are purely subjective, such states might become – in the face of a new piece of information – either unambiguously "impossible" or unambiguously "possible". The pessimistic updating permits both variants while the belief persistence axiom enforces the former. However, both variants are legitimate and which one will materialize solely depends on the Receiver's evaluation of the new piece of information he receives (e.g., ambiguous message).

In signaling games, relaxing the belief persistence axiom has a clear behavioral meaning. The pessimistic Receiver might be unable to learn the true type from the message observed in a separating equilibrium. On the contrary, in separating PBE, learning (i.e., ascribing probability 1) is enforced by Bayes' rule. However, learning is a strong behavioral assumption and its descriptive validity ought to be tested experimentally. As a matter of fact, the Receiver's learning is not necessary for the Sender to reveal her private information. Without learning, there will be DSEs that are still consistent with the PBE behavior. In other cases, the incapability of inferring the true type will give rise to separating behavior that can never be observed under PBE. A possible application of the DSE notion could be to investigate how robust a given separating PBE is to the Receiver's incapability of learning due to his pessimistic belief change.

Another relevant question is how "much" of the belief persistence axiom it is reasonable to abandon? Besides the descriptive scope of the DSE, one may require that the violations of belief persistence do not conflict with assumptions on players' rationality. For this reason, the notion of coherent DSE has been advocated. Under the notion of coherent beliefs, the Receiver only regards the types who are Choquet rational as being "possible". In a coherent DSE, the Receiver always infers the true type for whom sending the message on-the-equilibrium-path is the strictly dominant strategy. We believe that the coherent DSE is a more sound solution concept and it is worth of further exploration and application.

Appendix A. The DSEs for the signaling game of Ryan (2002a)

This appendix derives the DSEs for the game presented in Fig. 2.

Let p be a probability distribution $\{t_1, t_2\} \times \{L, R\}$ with $p(\{t_1, L\}) = p(\{t_2, R\}) = \frac{1}{2}$. Fix a partition $\{T_1, T_2\}$. Define $\{\nu^R, \{\nu^R_m\}_{m\in\mathcal{M}}\}$, the family of the Receiver's beliefs in the following way. For all $E \subseteq \mathcal{T} \times \mathcal{M}$,

$$\nu^R(E) = \rho p(E) + (1-\rho)\sum_{j=1}^{2} \beta_j(E)\, p(T_j), \quad 0 < \rho \leq 1,$$

$$\beta_j(E) = \begin{cases} 1 & \text{if } T_j \subseteq E, \\ 0 & \text{otherwise.} \end{cases}$$

$$\nu^R_L(E) = \begin{cases} \dfrac{1}{2-\rho} & \text{if } E = T_1, \\ \dfrac{1-\rho}{2-\rho} & \text{if } E = T_2, \end{cases}$$

$$\nu^R_R(E) = \begin{cases} \dfrac{1-\rho}{2-\rho} & \text{if } E = T_1, \\ \dfrac{1}{2-\rho} & \text{if } E = T_2. \end{cases}$$

The associated supports are:

$supp(\nu^R) = \{(t_1, L), (t_2, R)\}$,

$supp(\nu^R_L) = \{(t_1, L), (t_2, L)\}$,

$supp(\nu^R_R) = \{(t_1, R), (t_2, R)\}$.

Note that ν^R is an E-capacity and by Lemma 1, the types are unambiguous.

Case 1: $0 < \rho \leq \frac{2}{3}$. Define ν^S to be $\nu^S(D) = 1$ and $\nu^S(U) = 0$ for all messages. The associated support is $supp(\nu^S) = \{D\}$. Then, $[\nu^S, \nu^R, \{\nu^R_m\}_{m\in\mathcal{M}}]$ constitutes the DSE where the Receiver plays D after observing any message, and the Sender t_1 plays L and t_2 sends R. Ryan (2002a) assumed that $\rho = \frac{1}{2}$.

Case 2: $\frac{2}{3} \leq \rho \leq 1$. Let $\bar{\nu}^S$ by such that $\bar{\nu}^S(U) = 0$ and $\bar{\nu}^S(D) = 1$ for all messages. The associated support is $supp(\bar{\nu}^S) = \{U\}$. Then, $[\bar{\nu}^S, \nu^R, \{\nu^R_m\}_{m\in\mathcal{M}}]$ constitutes the DSE in which the Receiver plays U after observing any message, and the Sender t_1 plays L and t_2 sends R. Thus, for each $\rho > \frac{2}{3}$, the DSE supports the identical behavior of the unique separating PBE. Nevertheless, the Receiver's beliefs violate the belief persistence axiom whenever $\rho < 1$.

Appendix B. Pooling equilibrium with ambiguous types

This appendix illustrates a pooling DSE with ambiguous types that cannot be depicted by the standard PBE. Consider the signaling game in Fig. B.3.

In this game, for any $p_1 \in (0, 1)$ and $\alpha \in (0, \frac{1}{2})$, the PBE is either

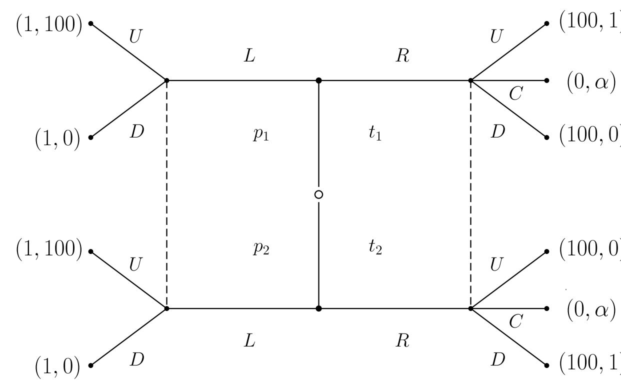

Fig. B.3. Pooling behavior under ambiguous types and messages.

Table B.4
The Receiver's ex-ante beliefs.

E	$\nu^R(E)$
$\{(t_1, L)\}$	α_1
$\{(t_2, L)\}$	α_2
$\{(t_1, R)\}$	0
$\{(t_2, R)\}$	0
$\{(t_1, L), (t_1, R)\}$	p_1
$\{(t_2, L), (t_2, R)\}$	p_2
$\{(t_1, R), (t_2, R)\}$	0
$\{(t_1, L), (t_2, L)\}$	ρ
$\{(t_1, L), (t_2, R)\}$	p_1
$\{(t_1, R), (t_2, L)\}$	p_2
$\{(t_1, L), (t_2, L), (t_1, R)\}$	ρ
$\{(t_1, L), (t_2, L), (t_2, R)\}$	ρ
$\{(t_1, R), (t_2, L), (t_2, R)\}$	p_2
$\{(t_1, L), (t_1, R), (t_2, R)\}$	p_1

$0 \leq \alpha_1 < p_1$ and $0 \leq \alpha_2 < p_2$
$\max\{p_1, p_2\} < \rho < 1$.

(i) Pooling on R; $[(R, R), (U, U), \mu(t_1 \mid L) = q \in [0, 1], \mu(t_1 \mid R) = p_1 > p_2]$, or
(ii) Pooling on R; $[(R, R), (U, D), \mu(t_1 \mid L) = q \in [0, 1], \mu(t_1 \mid R) = p_1 < p_2]$.

For the Sender, if she pools on message L, she always attains the payoff 1 regardless of the Receiver's response. However, as long as she believes that the Receiver will never play C after observing R, it is better for her to send message R giving her the payoff 100.

$$\int_{\mathcal{T}} u^R(R, r, t)\, d\nu^R_R = \begin{cases} 0 & \text{if } r \in \{U, D\}, \\ \alpha \in (0, \frac{1}{2}) & \text{if } r = C. \end{cases}$$

Conclusively, the Receiver's pessimism leads him to play C and the Sender who anticipates this pools on L.

For the Receiver, playing U after L is dominant action for any type of the Sender. Also, as long as the Receiver holds additive beliefs, playing C is never optimal after observing R. Thus, under the regime of additive beliefs, the Receiver will never choose response C even though it assures him the payoff $\alpha \in (0, \frac{1}{2})$. However, if the Sender believes that C will be the response to message R, then she should send L instead of R, which brings the safe payoff 1. The family of capacities presented in Tables B.4 and B.5 support this pooling DSE. The Receiver's capacity is additive on the types, yet, they are ambiguous.

Appendix C. Counter examples

Example 4. This example illustrates that Proposition 3 does not hold if the Receiver's equilibrium capacity is non-convex. Consider a signaling game with $|\mathcal{M}| = |\mathcal{T}| = 3$. Suppose that there exists

Table B.5
Conditional beliefs.

m	E	$\nu_m^R(E)$
L	T_1	p_1
	T_2	p_2
R	T_1	0
	T_2	0
m	E	$\nu_{(m)}^S(E)$
L	$\{U\}$	1
	$\{D\}$	0
R	$\{C\}$	1
	$\{U, D\}$	0

a separating DSE $\left[\nu^S, \nu^R, \{\nu_m^R\}_{m\in\mathcal{M}}\right]$ where ν^R is defined as follows: For every $T_j \in \mathcal{P}_{\mathcal{T}}$ and every $E \subseteq T_j$, let

$$\mu^R(E) := \begin{cases} \alpha_j & \text{if } E = \{(t_j, m_j)\}, \\ p(t_j) & \text{if } \{(t_j, m_j)\} \subset E, \\ 0 & \text{otherwise.} \end{cases}$$
$$0 < \alpha_j < p(t_j) \quad \text{for all } j \in \{1, 2, 3\}.$$

Now, for all $A \subseteq \mathcal{T} \times \mathcal{M}$, ν^R is defined as

$$\nu^R(A) = \sum_{j=1}^{3} \mu^R(A \cap T_j). \tag{C.1}$$

Notice that ν^R reveals that the Sender's types are unambiguous. The capacity is neither additive nor convex.[26] However, it respects the belief persistence axiom on every $m \in \mathcal{M}$. For every $j \in \{1, 2, 3\}$, the belief persistence axiom on m_j requires that $\nu_{m_j}^R(T_i) = 0$ for all $i \neq j$. Since $\nu^R(M_j^c) < 1$, $\nu_{m_j}^R(T_i) = 0$ if and only if $p(t_i) = \nu^R(T_i \cap M_j^c)$ which is satisfied by Eq. (C.1).

Example 5. This example shows that $|\mathcal{M}| = |\mathcal{T}|$ in Proposition 2 cannot be relaxed. Consider a signaling game with $|\mathcal{M}| = 3$ and $|\mathcal{T}| = 2$. Suppose that there exists a separating DSE, $[\nu^S, \nu^R, \{\nu_m^R\}_{m\in\mathcal{M}}]$, in which type t_1 sends m_1, and t_2 sends either m_2 or m_3. The Receiver's capacity ν^R is defined as follows: For every $j = 1, 2$, and every $E \subseteq T_j$, let

$$\mu^R(E) = \begin{cases} p(t_1) & \text{if } \{(t_1, m_1)\} \subseteq E \subseteq T_1, \\ \alpha_2 & \text{if } E = \{(t_2, m_2)\} \text{ or } E = \{(t_2, m_1), (t_2, m_2)\}, \\ \alpha_3 & \text{if } E = \{(t_2, m_3)\} \text{ or } E = \{(t_2, m_1), (t_2, m_3)\}, \\ p(t_2) & \text{if } E = \{(t_2, m_2), (t_2, m_3)\} \text{ or } E = T_2, \\ 0 & \text{otherwise.} \end{cases}$$
$$\alpha_2 + \alpha_3 < p(t_2), \quad 0 < \alpha_i, \ i = 2, 3.$$

And for $A \subseteq \mathcal{T} \times \mathcal{M}$, ν^R is defined as

$$\nu^R(A) = \sum_{j=1}^{2} \mu^R(A \cap T_j). \tag{C.2}$$

Notice that ν^R is convex and it reveals that the Sender's types are unambiguous. Moreover, ν^R respects the belief persistence axiom on every m and m^c. Notice that the belief persistence on M_1^c implies $\nu^R(T_1) = \nu^R(T_1 \cap M_1)$; on M_1 implies $\nu^R(T_2) = \nu^R(T_2 \cap M_1^c)$; and on M_j for $j = 2, 3$, implies $\nu^R(T_1) = \nu^R(T_1 \cap M_j^c)$. By construction of ν^R, all the conditions are respected. Yet, ν^R is non-additive.

[26] To verify the non-convexity of ν^R, take events $A = \{(t_1, m_1), (t_1, m_2)\}$ and $B = \{(t_1, m_1), (t_1, m_3)\}$. Hence, $p(t_1)+\alpha_1 = \nu^R(A\cup B)+\nu^R(B\cap B) < \nu^R(A)+\nu^R(B) = 2p(t_1)$.

Appendix D. Proofs

Lemma 0. *Let ν be a capacity on Σ and $\mathcal{P} = \{E_1, \ldots, E_I\}$ be a partition. Suppose that every $E \in \mathcal{P}$ is an unambiguous event. Then, for every $A \in \Sigma$:*

$$\nu(A) = \sum_{i=1}^{I} \nu(A \cap E_i). \tag{D.1}$$

Proof of Lemma 0. Fix an event $A \in \Sigma$. Since E_1 is unambiguous, one has

$$\nu(A) - \nu(A \cap E_1^c) = \nu(A \cap E_1). \tag{D.2}$$

Also, E_{k+1} is an unambiguous event for every $k = 1, \ldots, I - 1$. Thus, one has

$$\nu\Big(A \cap \big(\bigcup_{i=1}^{k} E_i\big)^c\Big) - \nu\Big(A \cap \big(\bigcup_{i=1}^{k+1} E_i\big)^c\Big) = \nu(A \cap E_{k+1}). \tag{D.3}$$

By summing all I equations, one obtains the following identity:

$$\nu(A) = \sum_{i=1}^{I} \nu(A \cap E_i). \quad \square \tag{D.4}$$

Proof of Lemma 1. Fix a partition $\mathcal{P} = \{E_1, \ldots, E_I\}$. Let $\nu_{p,\rho}(\cdot)$ be an E-capacity based on a probability distribution p on S and a parameter $\rho \in [0, 1]$. For an event $A \in \Sigma$, define

$$F_A := \{s \in S \mid s \in E_i \text{ and } E_i \subseteq A \text{ for some } i = 1, \ldots, I\}. \tag{D.5}$$

Then, $\nu_{p,\rho}(\cdot)$ can be written as

$$\nu_{p,\rho}(A) = \rho p(A) + (1 - \rho)p(F_A). \tag{D.6}$$

For every $E \in \mathcal{P}$, observe that $F_{A\cap E} \cap F_{A\cap E^c} = \emptyset$ and $F_A = F_{A\cap E} \cup F_{A\cap E^c}$. The former equation holds true since $(A \cap E) \cap (A \cap E^c) = \emptyset$. For the latter identity, one has the following: If $s \in F_A$, then there must exist $E_i \subseteq A$ such that $s \in E_i$. If $E = E_i$, $s \in F_{A\cap E}$; otherwise, $s \in F_{A\cap E^c}$. If $s \in F_{A\cap E}$, then $s \in E \subseteq A \cap E$. Thus, $s \in F_A$. If $s \in F_{A\cap E^c}$, then $s \in E_i \subseteq A \cap E_i$ for some i. Thus, $s \in F_A$.

Therefore, for all $E \in \mathcal{P}$, we have

$$\begin{aligned} \nu_{p,\rho}(A) &= \rho p(A) + (1 - \rho)p(F_A) \\ &= \rho p(A \cap E) + \rho p(A \cap E^c) \\ &\quad + (1 - \rho)p(F_{A\cap E} \cup F_{A\cap E^c}) \\ &= \rho p(A \cap E) + (1 - \rho)p(F_{A\cap E}) \\ &\quad + \rho p(A \cap E^c) + (1 - \rho)p(F_{A\cap E^c}) \\ &= \nu_{p,\rho}(A \cap E) + \nu_{p,\rho}(A \cap E^c), \end{aligned}$$

showing that E is unambiguous. $\square$

Proof of Theorem 1. Consider $U \in \Sigma$ and suppose that U is revealed to be unambiguous by $\succcurlyeq$. Fix a conditioning event $E \in \Sigma$. For any $f, g, k, k' \in \mathcal{A}$, the following is true

$$\begin{aligned} & fUk \succcurlyeq_E gUk && \text{(D.7)} \\ \iff & \big(fUk\big)E\,h \succcurlyeq \big(gUk\big)E\,h && \text{(D.8)} \\ \iff & \big(fE\,h\big)U\big(kE\,h\big) \succcurlyeq \big(gE\,h\big)U\big(kE\,h\big) && \\ \iff & \big(fE\,h\big)U\big(k'E\,h\big) \succcurlyeq \big(gE\,h\big)U\big(k'E\,h\big) && \text{(D.9)} \\ \iff & \big(fUk'\big)E\,h \succcurlyeq \big(gUk'\big)E\,h && \\ \iff & fUk' \succcurlyeq_E gUk'. && \text{(D.10)} \end{aligned}$$

(D.8) and (D.10) follow from the Definition of an h-Bayesian updating rule. Identity (D.9) follows from the assumption that U is revealed to be unambiguous by $\succcurlyeq$ and thus $\succcurlyeq$ satisfies the Sure-Thing

Principle constrained to U and U^c by Proposition 3.1 in Dominiak and Lefort (2011). Hence, by (D.7) and (D.10), we conclude that also the conditional preference $\succcurlyeq_E$ satisfies the Sure-Thing Principle constrained to U and U^c. That is, for any $f, g, k, k' \in \mathcal{A}$,

$$fUk \succcurlyeq_E gUk \iff fUk' \succcurlyeq_E gUk'. \quad \square$$

Proof of Lemma 2. Let ν be a capacity defined on $2^{\mathcal{T}\times\mathcal{M}}$ where $\mathcal{T} = \{t_j\}_{j=1}^J$ and $\mathcal{M}$ are finite sets. Suppose that every element in $\mathcal{P}_\mathcal{T} = \left\{\{t_j\} \times \mathcal{M} \mid t_j \in \mathcal{T}\right\}$ is unambiguous while the elements in $\mathcal{P}_\mathcal{M} = \left\{\mathcal{T} \times \{m\} \mid m \in \mathcal{M}\right\}$ might be ambiguous with respect to the ex-ante capacity ν. For the sake of brevity, let $T_j := \{t_j\} \times \mathcal{M}$ and $M := \mathcal{T} \times \{m\}$ denote the marginal events of $\mathcal{T} \times \mathcal{M}$.

Now, fix $M := \mathcal{T} \times \{m\}$, a conditioning event. By Theorem 1, every $T_j \in \mathcal{P}_\mathcal{T}$ is unambiguous with respect to the conditional capacity ν_M. Thus, for all $i \neq j$, one has

$$\begin{aligned}\nu_M(T_i \cup T_j) &= \nu_M\big((T_i \cup T_j) \cap T_j\big) + \nu_M\big((T_i \cup T_j) \cap T_j^c\big)\\ &= \nu_M(T_i) + \nu_M(T_j).\end{aligned}$$

By repeating this argument for all js, one gets

$$1 = \nu_M\left(\bigcup_{j=1}^J T_j\right) = \sum_{j=1}^J \nu_M(T_j).$$

Thus, ν_M is additive on $\mathcal{P}_\mathcal{T}$. By Theorem 3.2 in Gilboa and Schmeidler (1993), $\succcurlyeq_M$ admits a Choquet expected utility representation with respect to the unconditional utility function u and ν_M. Since ν_M is additive on $\mathcal{P}_\mathcal{T}$, the conditional preference $\succcurlyeq_M$ is SEU. $\square$

Proof of Proposition 1. First, we prove a lemma showing that the conditional capacity updated on the message on-the-equilibrium-path coincides with the prior probability distribution over the Sender's types. As a next step, we construct a PBE which is behaviorally equivalent to a fixed pooling DSE.

Lemma 3. *Let $\left[\nu^S, \nu^R, \{\nu_m^R\}_{m\in\mathcal{M}}\right]$ be a pooling DSE for a signaling game in Γ. and let m' be the message on-the-equilibrium path. Assume that the Sender's types are unambiguous. Then, $\nu_{m'}^R(T_j) = p(t_j)$ for all $j = 1, \ldots, J$.*

Proof of Lemma 3. Let $\left[\nu^S, \nu^R, \{\nu_m^R\}_{m\in\mathcal{M}}\right]$ be a pooling DSE. Assume, without loss of generality, that m_1 is the message on-the-equilibrium path. Let $supp(\nu^R) = \{(t_1, m_1), (t_2, m_1), \ldots, (t_J, m_1)\} := M_1$ be a DW-support associated with the equilibrium capacity ν^R of the pooling DSE. By conditioning on M_1, the Dempster–Shafer updating rule delivers

$$\nu_{m_1}^R(T_j) = \frac{\nu^R\big((T_j \cap M_1) \cup M_1^c\big) - \nu^R(M_1^c)}{1 - \nu^R(M_1^c)}$$
$$\text{for any } j = 1, \ldots, J.$$

Since M_1 is a DW-support, $\nu^R(M_1^c) = 0$. Thus, $\nu_{m_1}^R(T_j)$ is well defined and

$$\nu_{m_1}^R(T_j) = \nu^R\big((T_j \cap M_1) \cup M_1^c\big).$$

By the additive-separability condition of Assumption 2 and Lemma 0, one has

$$\begin{aligned}\nu_{m_1}^R(T_j) &= \sum_{j=1}^J \nu^R\Big(\big((T_j \cap M_1) \cup M_1^c\big) \cap T_j\Big),\\ &= \nu^R(T_j) + \sum_{k=1,k\neq j}^J \underbrace{\nu^R(T_k \cap M_1^c)}_{:=0},\\ &= \nu^R(T_j).\end{aligned}$$

By Assumption 1, $\nu^R(T_j) = p(t_j)$ is the probability of type t_j revealed by ν^R. Thus, $\nu_{m_1}^R(T_j) = p(t_j)$ for any $j = 1, \ldots, J$. $\square$

Let $\left[\nu^S, \nu^R, \{\nu_m^R\}_{m\in\mathcal{M}}\right]$ be the pooling DSE used in the proof above, and $supp(\nu^R) = \{(t_1, m_1), (t_2, m_1), \ldots, (t_J, m_1)\} := M_1$ be the DW-support associated with ν^R. By construction, we show that there exists an additive probability measure π^R on $\mathcal{T} \times \mathcal{M}$ satisfying

(i) $supp(\nu^R) = supp(\pi^R)$, and
$supp(\nu_m^R) = supp(\pi_m^R) \quad \forall m \in \mathcal{M}$,

(ii) $r^*(m) \in supp(\nu_{(m)}^S)$

$$\Rightarrow r^*(m) \in \arg\max_{r\in\mathcal{R}} \int_\mathcal{T} u^R(m, r, t) d\pi_m^R \quad \forall m \in \mathcal{M},$$

(iii) π_m^R is derived by Bayes' rule whenever possible.

Notice that $\nu_{m_1}^R$ is well defined since $\nu^R(M_1^c) = 0$. By Lemma 3, $\nu_{m_1}^R(T_j) = p(t_j)$ for any $j = 1, \ldots, J$, where $p(t_j)$ is the probability of type t_j revealed by ν^R (by Assumption 1). Now, construct π^R on $\mathcal{T} \times \mathcal{M}$ as follows. For every $j = 1, \ldots, J$:

$$\pi^R(\{(t_j, m_i)\}) = \begin{cases} p(t_j) & \text{if } i = 1,\\ 0 & \text{otherwise.}\end{cases}$$

Thus, $supp(\pi^R) = supp(\nu^R)$. Moreover, $\pi_{m_1}^R$ on M_1 derived by Bayes' rule is well defined and $\pi_{m_1}^R(T_j) = p(t_j)$ for every $j = 1, \ldots, J$. For any $m \neq m_1$, π_m^R is not well defined. By Lemma 2, ν_m^R is additive for any $m \in \mathcal{M}$ whenever ν_m^R is well defined. If, however, there is a message off-the-equilibrium path for which ν_m^R is not well defined, ν_m^R is chosen arbitrary. We make use of our assumption that the arbitrary conditionals are additive as well. Hence, we can set $\pi_m^R = \nu_m^R$ for any $m \neq m_1$. Thus, $supp(\pi_m^R) = supp(\nu_m^R)$ for every $m \in \mathcal{M}$ and Conditions (i), (ii) and (iii) above are satisfied. By our assumption, ν^S is a family of additive capacities $\{\nu_{(m)}^S\}_{m\in\mathcal{M}}$. Therefore, $[\nu^S, \pi^R, \{\pi_m^R\}_{m\in\mathcal{M}}]$ constitutes a PBE that is behaviorally equivalent to the given pooling DSE, $[\nu^S, \nu^R, \{\nu_m^R\}_{m\in\mathcal{M}}]$. Since the DSE is chosen arbitrarily, we conclude that the result holds true for any pooling DSE. $\square$

Proof of Proposition 2. Let $\left[\nu^S, \nu^R, \{\nu_m^R\}_{m\in\mathcal{M}}\right]$ be a separating DSE. By the definition of separating equilibrium, we may order types and messages so that, without loss of generality, the Sender sends m_j when her type is t_j with $j = 1, \ldots, J$. Let $supp(\nu^R) = \{(t_1, m_1), \ldots, (t_J, m_J)\}$ be a DW-support associated with the ex-ante capacity ν^R. By Assumption 1, $\nu^R(T_j) = p(t_j)$ denotes the probability of type j revealed by ν^R. By the definition of a DW-support, $\nu^R(\{s\}) = 0$ for any $s \notin supp(\nu^R)$. Thus, the Receiver's equilibrium capacity ν^R on the states in $\mathcal{T} \times \mathcal{M}$ can be expressed as follows

$$\nu^R\big(\{(t_j, m_k)\}\big) = \begin{cases} 0 & \text{for } j \neq k,\\ \alpha_j \in [0, p(t_j)] & \text{for } j = k.\end{cases} \tag{D.11}$$

Let $M_j^c := \mathcal{T} \times \{m_1, \ldots, m_{j-1}, m_{j+1}, \ldots, m_J\}$ be the event that messages other than m_j are sent. It will be shown that the belief persistence axiom satisfied on M_j and M_j^c implies that $\alpha_j = p(t_j)$ for all $j = 1, \ldots, J$.

Fix a type t_j with $j = 1, \ldots, J$. Let $supp(\nu_{m_j^c}^R)$ be a DW-support associated with the conditional capacity $\nu_{m_j^c}^R$. Since $supp(\nu^R) \cap M_j^c \neq \emptyset$, by the belief persistence axiom, $supp(\nu_{m_j^c}^R) = supp(\nu^R) \cap M_j^c$ implying that $\nu_{m_j^c}^R(T_j) = 0$. That is,

$$\nu_{m_j^c}^R(T_j) = \frac{\nu^R\left((T_j \cap M_j^c) \cup M_j\right) - \nu^R(M_j)}{1 - \nu^R(M_j)} = 0. \tag{D.12}$$

By Assumption 2, Lemma 0, and Eq. (D.11), it is true that

$$\begin{aligned}\nu^R(M_j) &= \sum_{k=1}^{J} \nu^R(M_j \cap T_k)\\ &= \nu^R(M_j \cap T_j) \le \nu^R(T_j) = p(t_j) < 1.\end{aligned} \tag{D.13}$$

Hence, $\nu^R_{m_j^c}$ is well defined. Thus, Eq. (D.12) holds true if, and only if,

$$\nu^R\left((T_j \cap M_j^c) \cup M_j\right) = \nu^R(M_j). \tag{D.14}$$

By Assumption 2, Lemma 0 and (D.11), Eq. (D.14) can be equivalently expressed as

$$\begin{aligned}\nu^R(T_j) + \sum_{k=1,k\neq j}^{J} \nu^R(T_k \cap M_j) &= \sum_{k=1}^{J} \nu^R(T_k \cap M_j)\\ \nu^R(T_j) &= \nu^R(T_j \cap M_j)\\ p(t_j) &= \alpha_j.\end{aligned}$$

Since t_j was chosen arbitrarily, we conclude that $\alpha_j = p(t_j)$ for every $j = 1, \dots, J$. Further, we have $\nu^R(E_j) = p(t_j)$ for any E_j such that $\{(t_j, m_j)\} \subseteq E_j \subseteq T_j$. Thus, by Assumption 2, ν^R is additive on $\mathcal{T} \times \mathcal{M}$ and the Receiver's ex-ante CEU preference with respect to ν^R is SEU at the given separating DSE. Since the given DSE was chosen arbitrarily, the statement holds true for any separating DSE. □

Proof of Proposition 3. Let $\left[\nu^S, \nu^R, \{\nu^R_m\}_{m\in\mathcal{M}}\right]$ be a separating DSE. By the definition of separating equilibrium, we may order types and messages so that, without loss of generality, the Sender sends m_j when her type is t_j with $j = 1, \dots, J$. Let $supp(\nu^R) = \{(t_1, m_1), \dots, (t_j, m_j)\}$ be a DW-support associated with the ex-ante capacity ν^R. By Assumption 1, $\nu^R(T_j) = p(t_j)$ denotes the probability of type j revealed by ν^R. By the definition of a *DW*-support, $\nu^R(\{s\}) = 0$ for any $s \notin supp(\nu^R)$. Thus, the Receiver's equilibrium capacity ν^R on the states in $\mathcal{T} \times \mathcal{M}$ can be expressed as follows

$$\nu^R\left(\{(t_j, m_k)\}\right) = \begin{cases} 0 & \text{for } j \neq k,\\ \alpha_j \in [0, p(t_j)] & \text{for } j = k.\end{cases} \tag{D.15}$$

It will be shown that the belief persistence axiom satisfied on every message $m \in \mathcal{M}$ together with the convexity of ν^R implies $\alpha_j = p(t_j)$ for all $j = 1, \dots, J$.

Fix a message m_k with $k \neq j$. Since the belief persistence axiom, we have that $\nu^R_{m_k}(T_j) = 0$. Thus, the Dempster–Shafer-updating rule delivers

$$\nu^R(T_j \cap M_k^c) = p(t_j). \tag{D.16}$$

Since k was chosen arbitrary, Eq. (D.16) holds true for all $k \neq j$.

Now, consider two distinct messages, m_k and m_l, where $l \neq k \neq j$. By Assumption 2, convexity of ν^R and Eq. (D.16), one has

$$\begin{aligned}&\nu^R\big(T_j \cap (M_k^c \cap M_l^c)\big) + \nu^R\big(T_j \cap (M_k \cup M_l)^c\big)\\ &\quad\ge \nu^R(T_j \cap M_k^c) + \nu^R(T_j \cap M_l^c)\\ &p(t_j) + \nu^R\big(T_j \cap (M_k \cup M_l)^c\big) \ge p(t_j) + p(t_j)\\ &\nu^R\big(T_j \cap (M_k \cup M_l)^c\big) \ge p(t_j).\end{aligned} \tag{D.17}$$

Since $T_j \cap (M_k \cup M_l)^c \subset T_j$, monotonicity of ν^R implies that

$$p(t_j) \ge \nu^R\big(T_j \cap (M_k \cup M_l)^c\big) \tag{D.18}$$

and thus by Eq. (D.16), it is true that

$$\nu^R\big(T_j \cap (M_k \cup M_l)^c\big) = p(t_j). \tag{D.19}$$

By repeating the same arguments for three distinct message m_k, m_l and m_m, where $k \neq l \neq m \neq j$, Eq. (D.19) implies that

$$\begin{aligned}&\nu^R(T_j) + \nu^R\big(T_j \cap (M_l \cup M_l \cup M_m)^c\big)\\ &\quad\ge \nu^R(T_j \cap (M_k^c \cup M_l^c)) + \nu^R(T_j \cap M_m^c)\\ &\nu^R\big(T_j \cap (M_k \cup M_l \cup M_m)^c\big) \ge p(t_j).\end{aligned} \tag{D.20}$$

Thus, by monotonicity of ν^R, it follows:

$$\nu^R\big(T_j \cap (M_k \cup M_l \cup M_m)^c\big) = p(t_j). \tag{D.21}$$

Since the set of messages is finite, one can consecutively apply convexity and monotonicity to all messages other than m_j, yielding the following identity:

$$\nu^R\Big(T_j \cap \big(\bigcup_{k=1,k\neq j}^{K} M_k\big)^c\Big) = p(t_j), \quad \text{where } J \ge K \ge 2. \tag{D.22}$$

Since $\left(\bigcup_{k\neq j} M_k\right)^c = M_j$, it is true that $\nu^R(T_j \cap M_j) = p(t_j)$ for all $j = 1, \dots, J$. By monotonicity of ν^R, we thus have that for all $j = 1, \dots, J$ and all $E_j \subseteq T_j$,

$$\nu^R\big(E_j\big) = \begin{cases} p(t_j) & \text{if } T_j \cap M_j \subseteq E_j,\\ 0 & \text{otherwise.}\end{cases} \tag{D.23}$$

Thus, the equilibrium capacity ν^R is additive on $\mathcal{T} \times \mathcal{M}$ and the Receiver's ex-ante CEU preference coincides with SEU at the given DSE. Since the DSE was chosen arbitrary, the result holds true at any separating DSE. □

Proof of Proposition 4. Let $\left[\nu^S, \nu^R, \{\nu^R\}_{m\in\mathcal{M}}\right]$ be a CDSE. Without loss of generality, suppose that m_1 is the strictly dominant message for the Sender's type t_1. Let $supp(\nu^R)$ be a DW-support associated with ν^R.

First, we show that $t_1 \notin supp(\nu^R_m)$ for all $m \neq m_1$; i.e., there exists no capacity ν^S such that $m \in B(t_1)$ (see Definition 10). Fix $m' \neq m_1$. Notice that the Choquet expected utility of m' w.r.t ν^R can be written as expected utility w.r.t the (rank-dependent) probability measure $\mu_{m'}$ (see Definition 1), i.e.,

$$\int_{\mathcal{R}} u^S(m', r, t)\mathrm{d}\nu^S_{(m')} = \int_{\mathcal{R}(m')} u^S(m', r, t)\mathrm{d}\mu^S_{m'}. \tag{D.24}$$

Since m' is strictly dominated by m_1 the following inequalities hold true:

$$\begin{aligned}&\min_{r\in\mathcal{R}} u^S(m_1, r, t_1) > \max_{r\in\mathcal{R}} u^S(m, r, t_1)\\ &\quad\ge \int_{\mathcal{R}(m')} u^S(m', r, t)\mathrm{d}\mu^S_{m'} \ge \min_{r\in\mathcal{R}} u^S(m', r, t_1).\end{aligned}$$

Thus, there cannot exist a capacity $\nu^S_{(m')}$ which rationalizes sending message m'. Hence, $m' \notin B(t_1)$ implying that $t_1 \notin supp(\nu^R_{m'})$. Since m_1 is the strictly dominant message for type t_1, $t_1 \notin supp(\nu^R_m)$ for all $m \neq m_1$ in the CDSE.

Now, it will be shown that, for all $E \subseteq T_1$,

$$\nu^R(E) = \nu^R(E \cap \{(t_1, m_1)\}) + \nu^R(E \cap \{(t_1, m_1)\}^c). \tag{D.25}$$

Since $\nu^R_{m_k}$ is additive (by Lemma 1) and $t_1 \notin supp(\nu^R_m)$ for all $m \neq m_1$, it holds true that $\nu^R_{m_k}(T_1) = 0$ for all $k = 2, \dots, J$.

Take $k \neq 1$. The Dempster–Shafer updating rule delivers,

$$\nu^R_{m_k}(T_1) = \frac{\nu^R((T_1 \cap M_k) \cup M_k^c) - \nu^R(M_k^c)}{1 - \nu^R(M_k^c)} = 0, \tag{D.26}$$

whenever $\nu^R_{m_j}$ is well defined (i.e., $\nu^R(M_j^c) < 1$). By the additive-separability condition of Assumption 2 and Lemma 0, Eq. (D.26) can be equivalently written as

$$\nu^R(T_1) + \sum_{j=2}^{J} \nu^R(M_k^c \cap T_j) = \sum_{j=1}^{J} \nu^R(M_k^c \cap T_j),$$
$$\nu^R(T_1) = \nu^R(M_k^c \cap T_1), \quad \text{for all } k \neq 1. \tag{D.27}$$

Consider two messages m_k and m_l where $k \neq l \neq 1$. By convexity of ν^R and (D.26), we obtain that

$$\nu^R\big((T_1 \cap M_k^c) \cup (T_1 \cap M_l^c)\big) + \nu^R\big(T_1 \cap (M_k \cup M_l)^c\big) \geq \nu^R(T_1 \cap M_k^c) + \nu^R(T_1 \cap M_l^c),$$
$$\nu^R\big(T_1 \cap (M_k \cup M_l)^c\big) \geq \nu^R(T_1).$$

Since $T_1 \cap (M_k \cup M_l)^c \subset T_1$, monotonicity of ν^R implies

$$\nu^R(T_1) \geq \nu^R\big(T_1 \cap (M_k \cup M_l)\big)^c.$$

Thus, for all $i \neq j \neq 1$, it is true that

$$\nu^R\big(T_1 \cap (M_k \cup M_l)^c\big) = \nu^R(T_1). \tag{D.28}$$

By repeating the same argument for another distinct $k \neq l \neq m \neq 1$, we get

$$\nu^R\big(T_1 \cap (M_k \cup M_l \cup M_m)^c\big) = \nu^R(T_1).$$

Thus, for all messages other than m_1, we have

$$\nu^R\Big(T_1 \cap \big(\textstyle\bigcup_{k=2}^{K} M_k\big)^c\Big) = \nu^R(T_1), \quad \text{where } J \geq K \geq 3. \tag{D.29}$$

Thus, the following identity holds true:

$$\nu^R(E) = \nu^R(T_1), \tag{D.30}$$

for any $E \subseteq T_1$ such that $\{(t_1, m_1)\} \subseteq E \subseteq T_1$. Now, we show that $\nu^R(E) = 0$ for any E such that $E \subseteq T_1 \setminus \{(t_1, m_1)\}$. Consider $A := \{(t_1, m_1)\}$ and $B := T_1 \setminus \{(t_1, m_1)\}$. By Eq. (D.30), $\nu^R(A) = \nu^R(T_1)$. By convexity of ν^R and the additive-separability condition of Assumption 2, we have

$$\nu^R(T_1) = \nu^R(A \cup B) \geq \nu^R(A) + \nu^R(B)$$
$$= \nu^R(T_1) + \nu^R(T_1 \setminus \{(t_1, m_1)\}),$$

which is true if, and only if, $\nu^R(T_1 \setminus \{(t_1, m_1)\}) = 0$. Thus, by monotonicity of ν^R, we have that $\nu^R(E) = 0$ for all $E \subseteq T_1 \setminus \{(t_1, m_1)\}$. By Assumption 1, $\nu(T_1) = p(t_1)$ is the probability of type 1 revealed by ν^R. Summing up, we have shown that, for any $E \subseteq T_1$:

$$\nu^R(E) = \begin{cases} p(t_1) & \text{if } (t_1, m_1) \in E, \\ 0 & \text{otherwise.} \end{cases} \tag{D.31}$$

Eq. (D.31) implies (D.25). By Eq. (D.25), the additive-separability condition of Assumption 2 and Lemma 0, it is true that, for any $A \subset \mathcal{T} \times \mathcal{M}$:

$$\nu^R(A) = \nu^R(A \cap T_1) + \sum_{j=2}^{J} \nu^R(A \cap T_j),$$
$$= \nu^R(A \cap \{(t_1, m_1)\}) + \nu^R(A \cap (T_1 \setminus \{(t_1, m_1)\}))$$
$$+ \sum_{j=2}^{J} \nu^R(A \cap T_j),$$
$$= \nu^R(A \cap \{(t_1, m_1)\}) + \nu^R(A \cap \{(t_1, m_1)\}^c).$$

This shows that ν^R is additively-separable across $\{(t_1, m_1)\}$ and $\{(t_1, m_1)\}^c$. Thus, $\{(t^o, m^o)\}$ is unambiguous. □

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